Transversally Elliptic Operators

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Abstract

In this paper we investigate the index theory of transversally elliptic pseudo-differential operators in the framework of noncommutative geometry.

We give examples of spectral triples in the sense of Alain Connes and Henri Moscovici in [12] that are transversally elliptic but non-elliptic. We prove that these spectral triples satisfy the conditions which ensure the Connes-Moscovici local index formula applies. We show that such a spectral triple has discrete dimensional spectrum.

We introduce an algebra $\Psi^\infty \rtimes G$ consisting of families of of pseudo-differential operators, but with convolution like product over parameter space G. We show the symbolic calculus of this algebra, which is similar to Ψ^∞, the algebra of pseudo-differential operators.

Given a spectral triple on $\Psi^\infty \rtimes G$, we show that there is a finite number of trace-like functionals $\tau_0, \tau_1, \ldots, \tau_N$ on $\Psi^\infty \rtimes G$ which are defined and used in [12] for the computation of the local index formula. Those τ-functionals generalize the Wodzicki residue of $\Psi^\infty = \Psi^\infty \rtimes \{1\}$, which is τ_0, the only nonzero among these τ functionals. Only the last nonzero τ_N is a trace on $\Psi^\infty \rtimes G$. It is shown N is bounded by the sum of the dimensions of the compact Lie group G, and the underlying manifold M on which G acts. Moreover those τ-functionals, evaluated at $A \in \Psi^\infty \rtimes G$, are determined by the transversal symbol of A.

The calculus on $\Psi^\infty \rtimes G$ is helpful in the computation of Connes' Chern character of a spectral triple. We show that Connes-Moscovici local index formula still works in the transversally elliptic case.

1 Introduction

1.1 Background

In this section we briefly review some historical background that motivated and influenced our work the most.

Atiyah introduced in a 1974 lecture note [4] the index of an invariant transversally elliptic pseudo-differential operator P relative to a compact Lie group G action on a compact manifold M. This index generalizes the character of the index of an invariant elliptic operator, which is a smooth, central function on G. Although the kernel and co-kernel of a transversally elliptic operator P are no longer finite dimensional as in the elliptic case, the index

$$index^G(\sigma(P)) = ker(P) - ker(P^*) \in C^{-\infty}(G)^G$$

makes sense as a central distribution on G. The index depends on the equivariant K-theory class of the transversal part of its principal symbol. An explicit index formula was given in [4] for torus acting with finite isotropy.

The operator algebra approach to the transversally elliptic operators followed soon afterwards. P. Julg (1982, [24]) gave the following observation. Just like an elliptic pseudo-differential operator gives a K-homology class in $KK(C(M), \mathbb{C})$, a transversally elliptic pseudo-differential operator naturally induces a K-homology class in $KK(A, \mathbb{C})$, where the algebra A is the crossed product algebra $C(M) \rtimes G$.

In his seminal 1985 paper [13], where noncommutative differential geometry was introduced, Alain Connes used smooth cross product algebras as important examples for noncommutative geometry. Transversally elliptic operators for foliations were studied as important examples of Fredholm modules over smooth groupoid algebras. The index of transversally elliptic operator problem evolves into the Connes-Chern character problem for summable Fredholm modules. The group action case and the foliation case are not identical although quite similar. The common ground between the two is when the group action is free. So it became natural to study the smooth crossed product algebra $\mathcal{A} = C^\infty(M) \rtimes G$. Both the foliation smooth algebra and the group action smooth algebra can be viewed as smooth groupoid algebras.

Smagin and Shubin (1987, [35]) introduced the formal zeta function constructed from the transversally elliptic symbols. They showed the use of the resolution of singularities of the phase function. Later Yuri Kordyukov has

studied Fredholm modules for group action algebra and spectral triples for Riemannian foliation algebra (see, [26], and [27]).

Connes and Moscovici (1995, [12]) gave the general local index formula for spectral triples, together with a new working example for hypo-elliptic operator. Later they showed many important applications. We intend to use this result to give an index formula for transversally elliptic operators on a smooth manifold relative to a compact Lie group action.

Also, we have been inspired also by some work done essentially in classical (instead of operator algebraic) approach. For instance, Helga Baum (1983, [8]) studied the transversal index of pseudo-Riemannian Dirac operators; also Berline and Vergne (1997, [9]) computed Chern characters of so called transversally elliptic good symbols.

1.2 The main results

1.2.1 The spectral triple

The noncommutative "space" that we consider, is the smooth crossed product algebra $\mathcal{A} = C^\infty(M \rtimes G)$, which is a Fréchet algebra with a natural topology. The elements of $\mathcal{A}$ are smooth functions (denoted by a, b) on $M \times G$, with the product

$$(a * b)(x, g) = \int_G a(x, h) b(h^{-1}x, h^{-1}g) d\mu(h).$$

$\mathcal{A}$ can be viewed as the smooth convolution algebra of the groupoid $M \rtimes G$ induced by the group action, which is the replacement for (the algebra of functions on) the quotient space M/G. This algebra has been shown to be closed under holomorphic functional calculus (see [32]).

Let D be a first order transversally elliptic pseudo-differential operator. $\mathcal{H}$ is usually the graded direct sum of L^2-sections of the complex vector bundles on which D acts as an unbounded self-adjoint operator. Let ϵ be the grading operator of $\mathcal{H}$. Further we assume that D^2 has scalar symbol. Although this assumption is rather strong, there are always examples of such operators, such as the Dirac operator associated to a general Clifford module and a Clifford connection. $\mathcal{A}$ acts upon $\mathcal{H}$ by extending the G-action ρ on the fibers of the complex bundles:

$$(\rho(a) \cdot s)(x) = \int_G a(x, g)(\rho(g)s)(g^{-1}x) dg.$$

Thus for any $a \in \mathcal{A}$, $\rho(a) \in B(\mathcal{H})$, $\rho(a)\epsilon = \epsilon\rho(a)$, and $D\epsilon = -\epsilon D$.

A pseudo-differential operator K on $\mathcal{H}$ is called transversally smoothing (to be denoted by $K \in \mathcal{K}_{\mathcal{A}}$) if for any $a \in \mathcal{A}$, $\rho(a)K$ is a smoothing operator (in particular trace class).

We first show that $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple in the following sense: (i) for all $a \in \mathcal{A}$, $[D, \rho(a)] \in B(\mathcal{H})$; (ii) for any $a \in \mathcal{A}$, $\rho(a)(1 + |D|)^{-1}$ is compact; (iii) there is a transversally smoothing operator $K \in \mathcal{K}_{\mathcal{A}}$ such that $|D| + K$ is invertible, and the inverse $(|D| + K)^{-1}$ is in the class $\mathcal{L}^{(p,\infty)}$.

Recall that for $p = \dim M > 1$, where $\mathcal{L}^{(p,\infty)}$ is the ideal of $B(\mathcal{H})$ consisting of those compact operators T whose *n-th* characteristic value satisfies

$$\mu_n(|T|) = O(n^{-1/p}). \tag{1.1}$$

When $p = 1$, although (1.1) is stronger than $T \in \mathcal{L}^{(1,\infty)}$, it is also satisfied.

1.2.2 Noncommutative residues

Throughout this and the next section, we assume we have fixed a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as described in the previous section.

From the algebra $\mathcal{A}$ and its representation on $\mathcal{H}$, we introduce the crossed product algebra of pseudo-differential operator and the group G:

$$\Psi_G = \Psi^\infty(E) \rtimes G = \bigcup_k \Psi^k(E) \rtimes G.$$

The operators in this algebra are families of pseudo-differential operators with parameter space G and convolution-like composition maps. We show that, similar to ordinary pseudo-differential operators, on Ψ_G there is a similar filtration (in terms of order k) and symbolic calculus.

There is a distribution kernel for any operator in Ψ_G. The wave front set of such a kernel has a certain form (see 3.4.1), just like the wave front set of a pseudo-differential operator, which has in a diagonal form (the micro-local property).

The motivation for introducing these operators is the computation of the Connes-Moscovici local index formula, and is described in section 1.2.3.

For the transversally elliptic D in the spectral triple, $|D|$ is only transversally elliptic. But as mentioned, we may add an G-invariant, transversally smoothing K to $|D|$ so that the replacement $|D|+K$ is invertible and elliptic. By simple wave front set computation, it is clear that K does not affect the

asymptotic trace integral formulas. So for the convenience of argument we may ignore K.

On the algebra Ψ_G, we study the noncommutative residues defined in Connes-Moscovici [12]. For $P \in \Psi^k(E) \rtimes G$, the zeta function

$$\zeta_{P,D}(z) = Trace(P\,|D|^{-2z}) \tag{1.2}$$

is initially defined and analytic on the half plane $\{z \in \mathbb{C} : 2Re(z) > k + \dim M\}$.

For our spectral triple, we define the *dimension spectrum relative to* Ψ_G as the minimal closed subset of $\mathbb{C}$ on the complement of which $\zeta_{P,D}$ can be extended to a holomorphic function for any $P \in \Psi_G$. Ψ_G contains $\mathcal{A}_D$ as defined by Connes and Moscovici. The dimension spectrum could be larger if we replace Ψ_G by one of its subalgebra.

Analysis of behavior the of the trace (1.2) amounts to some oscillatory integrals with a phase function related to the fixed point sets of the G action on M.

Theorem 1.2.1. *The dimension spectrum relative to* Ψ_G *is a discrete subset of the nationals* $\mathbb{Q}$.

For any $P \in \Psi^\infty(E) \rtimes G$ and $q = 0, 1, \dots$ the noncommutative residues τ_q, $q = 0, 1, \dots$ are defined as

$$\tau_q(P) = \tau_q^{|D|}(P) = Res_{z=0}(z^q \zeta_{P,D}), \tag{1.3}$$

that is, the residues of $z^q \zeta_{P,D}$ at $z = 0$. These residues may depend on the choice of D. For convenience, $\tau_{-1}(P)$ is also defined this way, it is the value of the zeta function at zero if zero is not a pole.

Theorem 1.2.2. *For any* $P \in \Psi_G$*, the poles of* $\zeta_{P,D}$ *are of multiplicity at most* $\dim M + \dim G$*. Therefore, there are only finitely many nonzero noncommutative residues* τ_q*,* $q = 0, 1, \dots,$ *up to possibly* $\dim M + \dim G - 1$.

The dimension spectrum and the maximal q (for τ_q to be nonzero) depends primarily on the action of G. It is well known that when G is the trivial group, where $\Psi_G = \Psi$ is the algebra of pseudo-differential operators, only τ_0 is nonzero and it is the Wodzicki residue. The Wodzicki residue extends any one of the Dixmier traces on the subalgebra of pseudo-differential operators of order $-\dim M$ to the algebra of all pseudo-differential operators.

It is shown in [12] that only the last τ_q is a trace. As shown in [12], when $q > 0$, τ_q always vanishes on the Dixmier trace class operators (that is, $\mathcal{L}^{(1,\infty)}$) and in particular, on $\Psi^{-\dim M} \subset \mathcal{L}^{(1,\infty)}$. But on $\Psi_G^{-\dim M}$, $\tau_0(P)$ is equal to any Dixmier trace. Therefore τ_0 still extends any Dixmier trace as a functional but it may not be a trace on Ψ_G.

Theorem 1.2.3. *For any $q = 0, 1, \dots, \dim M + \dim G - 1$, τ_q vanishes on $\mathcal{K}_G$, the ideal of transversally smoothing operators.*

Theorem 1.2.4. *For any $q = 0, 1, \dots, \dim M + \dim G - 1$, and any $P \in \Psi_G$, τ_q depends only on the transversal part of the full symbols of P of order no lower than $-\dim M$.*

The above theorem asserts that the noncommutative residues are in principle "computable" in terms of symbolic calculus.

1.2.3 Connes-Moscovici local index formula

The index formula of $(\mathcal{A}, \mathcal{H}, D)$ follows from the Connes-Chern character formula in periodic cyclic cohomology. The operator-theoretical version of the local index formula is given in Connes-Moscovici [12] in full generality. With the the theorems in section 1.2.2 we show that it is possible to apply the Connes-Moscovici local index formula for our specific spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

For any operators $A \in \mathcal{A}$, we use the following notations

$$dA = [D, A],\ \nabla(A) = [D^2, A],\ A^{(k)} = \nabla^k(A).$$

Then all the above operators are in $\Psi_G = \Psi^\infty(E) \rtimes G$. In particular, the operators

$$a^0 (da^1)^{(k_1)} \dots (da^n)^{(k_n)} \tag{1.4}$$

are in $\Psi^\infty(E) \rtimes G$, where $a^0, \dots, a^n \in \mathcal{A}$, acting on $\mathcal{H}$ by ρ. Let $\mathcal{A}_D$ be the subspace of $\Psi^\infty(E) \rtimes G$ generated by those operators in (1.4).

$|D|^{-1}$ (which is really $(D + K)^{-1}$), is in $\Psi^{-1}(E) \rtimes G$ and we have

$$a^0 (da^1)^{(k_1)} \dots (da^n)^{(k_n)} \, |D|^{-2|k|-n} \in \Psi^0(E) \rtimes G. \tag{1.5}$$

In particular, the left hand side of (1.5) is bounded. So we assert the Connes-Moscovici local index formula holds.

Theorem 1.2.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be an even spectral triple defined by a first order transversally elliptic pseudo-differential operator D and with all the above conditions.*

The Connes character $ch(\mathcal{A}, \mathcal{H}, D)$ in periodic cyclic cohomology is represented by the following even cocycle in the periodic cyclic cohomology:

$$\begin{aligned}\phi_{2m}(a_0, \ldots, a_{2m}) = \sum_{k \in \mathbb{Z}^{2m}, q \geq 0} & c_{2m,k,q} \cdot \\ & \tau_q \left(a^0 (da_1)^{(k_1)} \cdots (da_{2m})^{(k_{2m})} |D|^{-2|k|-2m} \right)\end{aligned} \tag{1.6}$$

for $m > 0$ and

$$\phi_0(a^0) = \tau_{-1}(\gamma a^0). \tag{1.7}$$

In the above formula $k = (k_1, \ldots, k_{2m})$ are multi-indices and $c_{2m,k,q}$ are universal constants given by

$$c_{2m,k,q} = \frac{(-1)^{|k|}}{k! \tilde{k}!} \sigma_q(|k| + m), \tag{1.8}$$

where $k! = k_1! \ldots k_{2m}!$, $\tilde{k}! = (k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \ldots + k_{2m} + 2m)$, and for any $N \in \mathbb{N}$, $\sigma_q(N)$ is the q-th elementary polynomial of the set $\{1, 2, \ldots, N - 1\}$.

There is a similar theorem for odd spectral triples.

1.3 Outline of the paper

In section 2 we review mostly known facts we need, including wave front set and pseudo-differential operators. Simple wave front set calculation allows us to use the spectral analysis of elliptic operators instead of transversally elliptic ones. In section 3 we first review Atiyah's definition of a transversally elliptic operator operators. Then we define the crossed product algebra $\mathcal{A}$, our noncommutative space. Next the algebra of crossed products with pseudo-differential operators is introduced. In the rest of this section we establish basic rules of calculus as needed later. The notion of transversally smoothing operators is introduced to simplify arguments. In section 4 we discuss trace formulas. First we study the trace class operators and the trace formula in terms of the distributional kernel. Then we show how the asymptotic expansion of oscillatory integrals comes into the picture. Next

we review some known theorems about the asymptotic expansion of oscillatory integrals. The definition noncommutative residue is reviewed and then their properties are discussed. This allows us to develop our main statements about the noncommutative residues. Further techniques are needed to decide these residues by the symbol. In section 5 we discuss the K-theoretic aspect of the index of the transversally elliptic operators. Here we show that the index of a transversally elliptic operator as defined by Atiyah [4] is a natural part of the K-theory of the noncommutative space $\mathcal{A}$. Next we show that the Connes-Chern character in K-theory is related to the transversal index. Finally we apply our residue formula to the Connes-Moscovici local index formula for the spectral triples decided by transversally elliptic operators.

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2 Preliminaries

2.1 Conventions

Unless otherwise specified, we use the following notations throughout the paper.

Let G be a compact Lie group acting smoothly on a compact smooth manifold M. We denote by $\mathfrak{g}$ the Lie algebra of G. For convenience, we assume that M is a compact, oriented Riemannian manifold with an invariant Riemannian metric and G acts by orientation preserving isometries. We also assume that a fixed G-invariant volume form on M is selected. For an element $g \in G$, let $M^g = M(g)$ be the fixed point set of g. (which under all the above assumptions is a disjoint union of totally geodesic submanifolds of M). We will consider complex hermitian G-equivariant bundles over M, which are assumed here to have smooth hermitian metrics preserved by the G-action.

Let $\mathbb{N}$ be the set of the natural numbers, including zero.

For a smooth manifold, let $C_c^\infty(M)$ be the compact supported, complex valued test functions with the induced limit topology of local test func-

tions (often denoted by $\mathcal{D}(M)$), and $C^{-\infty}(M)$ the corresponding distributions(often denoted by $\mathcal{D}'(M)$). We work mostly on compact manifolds, though. For a hermitian bundle E over M, $\Gamma_c^\infty(E)$ (or $\mathcal{D}(M,E)$) and $\Gamma^{-\infty}(E)$ (or $\mathcal{D}'(M,E)$) the sections of test functions and distributional sections.

For general notations or definitions used in this paper, the standard reference is Connes' book [14]. We refer to papers [5] and [6] and books [21] for other definitions and backgrounds.

2.2 Wave front set

2.2.1 Definition

For a distribution u with compact support to equal a smooth function, its Fourier transform needs to satisfy: for any $N \in \mathbb{N}$ there is a C_N such that

$$|\hat{u}(\xi)| \leq C_N(1+|\xi|)^{-N} \tag{2.1}$$

for all $\xi \in \mathbb{R}^n$.

Definition 2.2.1. Let u be a distribution on $\mathbb{R}^n$ with compact support. Then at $x \in \mathbb{R}^n$ its wave front set Σ_x is a cone of all $\eta \in \mathbb{R}^n \backslash \{0\}$ having no conic neighborhood V such that (2.1) is true for all $\xi \in V$. For an arbitrary distribution u on an open subset X of $\mathbb{R}^n$ let

$$\Sigma_x(u) = \bigcap_{\phi \in C_c^\infty(\mathbb{R}^n), \phi(x) \neq 0} \Sigma_x(\phi u). \tag{2.2}$$

The wave front set for u is

$$WF(u) = \{(x,\xi) \in X \times (\mathbb{R}^n \backslash \{0\}); \xi \in \Sigma_x(u)\}. \tag{2.3}$$

For a smooth vector bundle E over M, let $\Omega(M)$ be the volume bundle over M and $E' = Hom(E, \Omega(M))$. By definition, a distributional section is an element of $\Gamma^{-\infty}(M,E)$, the dual space of $\Gamma_c^\infty(M,E')$.

The invariant definition for vector-valued distributions is given below. For details, see, for example, [21], [16] or [31].

Definition 2.2.2. For a smooth bundle E on M, let $u \in \Gamma^{-\infty}(M,E)$ be a distributional section of E. The wave front set $WF(u)$ is a subset of $T^*M \backslash \{0\}$, such that for any $(x,\xi) \in T^*M \backslash \{0\}$, $\xi_x \notin WF(u)$ if and only if

for any (phase function) $\psi \in C_c^\infty(M \times \mathbb{R}^p, \mathbb{R})$, $d\psi(.,y)_x = \xi_x$, there is an $s \in \Gamma_c^\infty(M, E')$, $s(x) \neq 0$, and a neighborhood U_y of y in $\mathbb{R}^p$, such that for all $n \in \mathbb{N}$, and all $y \in U_y$,

$$|\langle u, exp(-it\psi(x, y'))s\rangle| = O(t^{-n}) \tag{2.4}$$

when $t \to \infty$.

Wave front set is a refinement of the singular support for distributions.

Proposition 2.2.3. *$WF(u)$ is a closed conic subset. The projection of $WF(u)$ onto M is the singular support of u.*

2.2.2 Push-forward of a distribution and its wave front set

For distributions, the push-forward is a natural operation, and the wave front set under this operation is described below.

Definition 2.2.4. Let $f : M \to N$ be a proper smooth map, Let S be a conic subset of $T^*M \backslash \{0\}$, then the push-forward f_*S of S is

$$f_*S = \{(y, \eta) \in T^*N\backslash\{0\} : \exists x \in M, y = f(x), (x, f^*(\eta)) \in (S \cup 0_{T^*M})\}. \tag{2.5}$$

Theorem 2.2.5. *Let $f : M \to N$ be a smooth map, $u \in C^{-\infty}(M)$ such that $f|_{supp(u)}$ is proper. Then the push-forward $u \mapsto f_*(u)$ is a well-defined continuous linear map*

$$f_* : C^{-\infty}(M) \to C^{-\infty}(N). \tag{2.6}$$

Moreover,

$$WF(f_*u) \subset f_*(WF(u)). \tag{2.7}$$

Example 2.2.6. (1) Take $M = \{p\}$, the 0-dimensional manifold of a single point, f is simply decided by its image $y = f(p)$ in N. Let 1_M be the constant function on M with value 1. Then $f_*(1_M)$ is simply δ_y, the delta function of N at the point y. Simple computations right from definitions show that $WF(\delta_y) = T_y^*N\backslash\{0\}$, $WF(1_M) = \varnothing$, and $f_*(\varnothing) = T_y^*N\backslash\{0\}$.

(2) A little more generally, let f be an embedding, let α be a compactly supported smooth function on M, $f_*(\alpha) = \alpha\mu_M$ is usually called the density of M in N, direct computation shows

$$WF(f_*\alpha) = \{(x, \xi) \in T^*N, x \in supp(\alpha), \xi \in \mathcal{N}_M^*(N)\backslash\{0\}\}, \tag{2.8}$$

where $\mathcal{N}^*_M(N) = ker(f^*)$ is the conormal bundle of M in T^*N. Compare $WF(\alpha) = \varnothing$ and $f_*(\varnothing) = ker(f^*)$.

(3) Let $\pi : M \times N \to N$ be the projection to N. Let M be compact so the push-forward of π_* is well-defined. For $v \in C^\infty_c(N)$ and $u \in C^{-\infty}(M \times N)$,

$$(\pi_* u, v) = (u, \pi^* v) = (u, 1_M \otimes v) = \int_M (u(x,\cdot), v(\cdot))dx, \tag{2.9}$$

the last formal integral is true when u is smooth. $WF(\pi_* u)$ is the projection of $WF(u)$ onto T^*N.

In fact, the push-forward operation contains the integration for a distribution as a special case. Let $N = \{pt\}$ be a single point manifold so the functions and distributions on it are just complex numbers. Let c be the unique map $M \to N$, $c_*(u)$ is defined when M is compact and it is just the number

$$c_*(u) = (u, 1_M) = \int_M u(x)dx. \tag{2.10}$$

where the last integral is formal.

(4) Now let $f : M \to N$ be a submersion from a compact manifold M to compact manifold N. Let u be a smooth function on M, so $WF(u) = \varnothing$. Then the push-forward $f_* u$ is still an integration along the vertical fibers with our chosen volume form.

$$WF(f_* u) = f_*(WF(u)) = \varnothing, \tag{2.11}$$

so as we know $f_* u$ is smooth. When u is a distribution $f_*(WF(u))$ is again the projection along vertical directions.

It is straightforward to extend the above theorem to maps between bundles, because the wave front set is a local concept.

Theorem 2.2.7. *Let $f : (E, M) \to (F, N)$ be a smooth bundle map between two hermitian bundles, $u \in \Gamma^{-\infty}(E)$ such that $f|_{supp(u)}$ is proper. Then the push-forward $u \mapsto f_*(u)$ is a well-defined continuous linear map*

$$f_* : \Gamma^{-\infty}(E, M) \to \Gamma^{-\infty}(F, N). \tag{2.12}$$

Moreover,

$$WF(f_* u) \subset f_*(WF(u)). \tag{2.13}$$

2.2.3 Schwartz kernel and wave front relation

For two locally convex topological vector spaces V and W, we denote by

$$\mathcal{L}_S(U, V) \tag{2.14}$$

the topological vector space of continuous linear maps from V to W, with the strong topology.

Let (E_1, M_1), (E_2, M_2) be two hermitian bundles, and let

$$A : \Gamma^{-\infty}(E_1) \to \Gamma^{-\infty}(E_2) \tag{2.15}$$

be a linear and continuous operator. There is a distributional section

$$K_A \in \Gamma^{-\infty}(E_2 \boxtimes E_1') \tag{2.16}$$

such that for any $u \in \Gamma^{-\infty}(E_1)$ and $v \in \Gamma^{-\infty}(E_2')$

$$(K_A, v \otimes u) = (v, Au). \tag{2.17}$$

We recall the Schwartz kernel theorem:

Theorem 2.2.8. *There is a unique continuous linear map*

$$\mathcal{L}_S(\Gamma_c^{-\infty}(E_1), \Gamma^{-\infty}(E_2)) \to \Gamma^{-\infty}(E_2 \boxtimes E_1') \tag{2.18}$$

that maps A to K_A such that (2.17) holds. This map is an isomorphism of the topological vector spaces.

Definition 2.2.9. The *wave front relation* of A is defined as

$$WF'(A) = \{(\eta_y, \xi_x) \in T^*(M_2 \times M_1) | (\eta_y, -\xi_x) \in WF(K_A)\}. \tag{2.19}$$

And we define $WF'_{M_i}(A)$, called $WF'(A)$'s projection on T^*M_i, $i = 1, 2$, as follows:

$$WF'_{M_1}(A) = \{(x, \xi_x) \in T^*M_1 \backslash \{0\} : \exists y \in M_2, (y, x, 0_y, \xi_x) \in WF'(A)\}, \tag{2.20}$$

and

$$WF'_{M_2}(A) = \{(y, \eta_y) \in T^*M_2 \backslash \{0\} : \exists x \in M_1, (y, x, \eta_y, 0_x) \in WF'(A)\}. \tag{2.21}$$

Note: In the event when $M_1 = M_2 = M$ (which we encounter in this paper), to avoid any confusion, we use an extra label, that is, $WF'_{M,1}$ and $WF'_{M,2}$ respectively.

Let (E_i, M_i), $i = 1, 2, 3$ be hermitian bundles over compact manifolds:

$$A : \Gamma^{\infty}(E_1) \to \Gamma^{-\infty}(E_2) \tag{2.22}$$

$$B : \Gamma^{\infty}(E_2) \to \Gamma^{-\infty}(E_3) \tag{2.23}$$

are linear and continuous operators.

Theorem 2.2.10. *If* $WF'_{M_2}(A) \cap WF'_{M_2}(B) = \varnothing$*, then*

$$B \circ A : \Gamma^{\infty}(E_1) \to \Gamma^{-\infty}(E_3) \tag{2.24}$$

is a well defined, linear and continuous operator. Moreover,

$$\begin{aligned} WF'(B \circ A) \subset\, & WF'(B) \circ WF'(A) \\ & \cup (WF'(B)_{M_3} \times 0_{T^*M_1}) \cup (0_{T^*M_3} \times WF'(A)_{M_1}). \end{aligned} \tag{2.25}$$

(By definition WF' *is a binary relation between* T^*M_2 *and* T^*M_1 *as sets, the composition "*$\circ$*" between wave front relations is that of the binary relation between sets.)*

We will need the following obvious corollary. Assume now $(E_i, M_i) = (E, M)$ are all the same.

Corollary 2.2.11. *If* $WF'_{M,2}(A) \cap WF'_{M,1}(B) = \varnothing$ *and* $WF'(B) \circ WF'(A) = \varnothing$*, then* $B \circ A$ *is a smoothing operator.*

Theorem 2.2.12. *The wave front relation for the kernel of* f_* *in theorem 2.2.7 satisfies*

$$WF'(f_*) \subset \{(\xi_x, \eta_{f(x)}) \in T^*(M \times N)\backslash\{0\} : f^*(\eta_{f(x)}) = -\xi_x \text{ or } f^*(\eta_{f(x)}) = 0\}. \tag{2.26}$$

Proof. We apply theorem 2.2.7 to the embedding $M \to M \times N$, $x \mapsto (x, f(x))$. ∎

2.3 Pseudo-differential operators

In this section we review some selected, basic but important facts about pseudo-differential operators, see [22] for details. In particularly we need to specify a Fréchet topology on the space of pseudo-differential operators. This is necessary since we need to integrate over the group G of a family of pseudo-differential operators, together with a continuous group action on such families. In [5] and [6], a good topology of pseudo-differential operators are given. Although for most of what we do, the topology on Op^m as in [6] is sufficient, the Fréchet topology decided by the symbol is more convenient for our purpose. Such a Fréchet topology has been well known in the early of symbols and pseudo-differential operators (see, for example, [37] and [15]).

2.3.1 Local calculus

Recall the definition of scalar symbols of order m on $\mathbb{R}^n$ (18.1.1 in [22]). We omit the space in the notations only when it is $\mathbb{R}^n$.

Definition 2.3.1. For any real number m, $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is defined as the set of all $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the following condition: for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ there is a positive constant $C_{\alpha,\beta}$ such that

$$p_{\alpha,\beta}(a) = \sup_{x,\xi} \left| \frac{\partial_\xi^\alpha \partial_x^\beta a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} \right| \leq C_{\alpha,\beta} \tag{2.27}$$

for all x, $\xi \in \mathbb{R}^n$.

S^m is a Fréchet space with semi-norms $p_{\alpha,\beta}$ for all possible multi-indices α and β in $\mathbb{N}^n$. Each symbol $a \in S^m$ gives rise to a pseudo-differential operator denoted by $Op(a) : \mathcal{S} \to \mathcal{S}$, acting on Schwartz functions $\mathcal{S}$ on $\mathbb{R}^n$: for $u \in \mathcal{S}$ and $x \in \mathbb{R}^n$,

$$Op(a)(u)(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)d\xi, \tag{2.28}$$

a is called the full symbol of $Op(a)$. This action extends to a continuous linear operator from $\mathcal{S}'$ onto itself. The action of $Op(a)$ is faithful, that is, $Op(a) = 0$ only when $a = 0$.

Definition 2.3.2. We define $\Psi^m = \Psi^m(\mathbb{R}^n)$ as the topological space of all operators on $\mathcal{S}'$ of the form $Op(a)$ for $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, with topology that of S^m. For any m, Ψ^m is a Fréchet space.

Theorem 2.3.3. *If $a_j \in S^{m_j}$, $j = 1, 2$, then there exists $b \in S^{m_1+m_2}$ such that*

$$Op(a_1)Op(a_2) = Op(b). \tag{2.29}$$

The symbol b is given by

$$b(x,\xi) = e^{i\langle D_y, D_\eta\rangle} a_1(x,\eta)a_2(y,\xi)|_{\eta=\xi, y=x} \tag{2.30}$$

and it has the following asymptotic expansion:

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a_1(x,\xi) D_x^\alpha a_2(x,\xi)|_{\eta=\xi, y=x} \tag{2.31}$$

where we recall the notation D_y^α means $(-i)^{|\alpha|}\partial_y^\alpha$.

Moreover, the product map

$$\begin{aligned} \Psi^{m_1} \times \Psi^{m_2} &\to \Psi^{m_1+m_2} \\ Op(a_1) \times Op(a_2) &\mapsto Op(a_1)Op(a_2) \end{aligned} \tag{2.32}$$

is jointly continuous.

For the proof, we refer to the proof of 18.1.8 in [22]. As a consequence, for $m \leq 0$, Ψ^m is a Fréchet algebra. Obviously, for $m_1 < m_2$, Ψ^{m_1} embeds naturally into Ψ^{m_2}. The union of all Ψ^m over all $m \in \mathbb{R}$ is denoted by Ψ^* or Ψ^∞, with the induced limit topology, it is also a Fréchet algebra.

Theorem 2.3.4. *Let $\kappa : X \to X_\kappa$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Then for any $a \in S^m$ there exists $a_\kappa \in S^m$ such that*

$$Op(a_\kappa) = \kappa_* \circ Op(a) \circ \kappa^* \tag{2.33}$$

This is the invariance of $\Psi^m(\mathbb{R}^n)$ under change of coordinates (see 18.1.17 of [22]).

2.3.2 The Fréchet Topology

The notion of pseudo-differential operators extend to manifolds.

Definition 2.3.5. Let M be a smooth manifold, a continuous operator:

$$P : C_c^\infty(M) \to C^\infty(M) \tag{2.34}$$

is called a pseudo-differential operator ($P \in \Psi^m(M)$) if for any differentiable chart of M:

$$\chi : U \subset M \to \chi(U) \subset \mathbb{R}^n$$

and for any $\phi, \psi \in C^\infty(M)$ with supports within U, we have

$$P_{\chi,\phi,\psi} = (\chi_*)(\phi P \psi)(\chi^*) \in S^m(\chi(U)). \tag{2.35}$$

Definition 2.3.6. We give $\Psi^m(M)$ the topology induced by all the semi-norms depending on ϕ and ψ,

$$\{p_{\alpha,\beta,\phi,\psi}(P) = p_{\alpha,\beta}(\phi P \psi)\}.$$

Proposition 2.3.7. *For a chosen compatible atlas of charts on a manifold M, let ϕ_i be a locally finite partition of unity subordinate to the covering induced by this atlas. Then*

$$\{p_{\alpha,\beta,\phi_i,\phi_j}(P) = p_{\alpha,\beta}(\phi_i P \phi_j)\}.$$

generate the topology of $\Psi^m(M)$. Consequently, $\Psi^m(M)$ is a Fréchet space.

Proof. We need to show that any semi-norm $p_{\alpha,\beta,\phi,\psi}$ is bounded by a finite sum of other semi-norms. By the partition of unity,

$$P = \sum_{i,j} \phi_i P \psi_j.$$

For any pair ϕ and ψ only finitely many of ϕ_i's will be needed in the partition of unity. This allows us to prove this proposition with the assumption that $P = \phi_i P \phi_j$. In other words, we can assume there is one chart and P has small enough support.

Note that multiplication by ψ is a pseudodifferential operator of order zero, that is $\psi \in S^0$, with symbol $\psi(x,\xi) = \psi(x)$. Using theorem 2.3.3 for ϕP and ψ, especially the joint continuity, we conclude the semi-norm $p_{\alpha,\beta,\phi,\psi}$ is bounded by a finite sum of semi-norms of ϕP, times some positive constants depending only on a finite number of semi-norm of ψ in S^0.

Then we repeat this argument with ϕP with $\phi \in S^0$ and P. It is routine to check that $\Psi^m(M)$ is complete. With countable semi-norms generating the topology, it is a Fréchet space. $\square$

Definition 2.3.8.

$$P : \Gamma_c^\infty(M, E) \to \Gamma^\infty(M, F) \tag{2.36}$$

is called pseudo-differential (to be denoted by $P \in \Psi^m(M; E, F)$) if for a given pair of trivializations φ_E of E and φ_F of F on $U \subset M$, and for all $u \in \Gamma_c^\infty(M, E)$, there exists $A_{ij} \in \Psi^m(U)$ such that

$$\varphi_F(Pu)_i = \sum A_{ij}(\varphi_E u)_j. \tag{2.37}$$

and the topology on $\Psi^m(M; E, F)$ over U is given by the union of semi-norms for A_{ij} over indices i, j on U.

It is straightforward to check that the definition does not depend on the choice of trivialization of E or F; The topology on $\Psi^m(M; E, F)$ is well defined; and $\Psi^m(M; E, F)$ is a Fréchet space.

Let $\Omega(M)$ be the volume bundle over M. Recall that, as in [5] and [6], E' is defined as $E^* \otimes \Omega(M)$. In [22], Hörmanderused the bundle $\Omega^{1/2}(M)$ of half densities to define the adjoint of a pseudo-differential operator. With the assumption that M is oriented, Ω and hence $\Omega^{1/2}$ are trivial bundles, so they may be omitted up to an bundle isomorphism decided by a chosen volume form. Hörmander's approach is equivalent to the one we adopt here. However this identification has a minor effect on the symbol.

$\Psi^m(M; E, E)$ is closed under $*$ and so becomes a $*$-algebra. we will use the abbreviation $\Psi^m(E) = \Psi^m(M; E, E)$.

In particular, for $m \leq 0$, $\Psi^m(E)$ is a Fréchet $*$-subalgebra.

2.3.3 Continuity under a Lie group action

Theorem 2.3.9. *Let G be a Lie group acting on M. For $g \in G$ and $P \in \Psi^m(M; E, F)$, let*

$$g(P) = g_*(P) = g \circ P \circ g^{-1} \tag{2.38}$$

where in the right hand side g acts on $\Gamma_c^\infty(M, E)$ and $\Gamma^\infty(M, E)$ by push-forward. Then this defines a continuous action G on $\Psi^m(M; E, F)$.

Proof. By the topology we choose, it is sufficient to prove the local version: without loss of generality, we assume $P = a(x, D)$ is supported in an open subset X of $\mathbb{R}^n$. Let κ_n be a sequence of elements in G converging to the identity, we may assume all the X_{κ_n} is contained in a single open subset X' of $\mathbb{R}^n$.

We need to show that under any semi-norm $p_{\alpha,\beta}$,

$$p_{\alpha,\beta}(\kappa_n(P)) \to p_{\alpha,\beta}(P)$$

In the proof of the local invariance theorem 2.3.4 in [22] (theorem 18.1.17 there). We choose $\phi \in C_c^\infty(X')$, such that $\phi(x) = 1$ for any x in the support of any $\kappa_n(P)$.

For any $\kappa = \kappa_n$ it is shown that

$$a_\kappa(x,\eta) = \phi(x)e^{-i\langle\kappa(x),\eta\rangle}a(x,D)(\phi(x)e^{i\langle\kappa(x),\eta\rangle} \tag{2.39}$$

is in S^m and it is implicit in the proof that for all $\alpha, \beta \in \mathbb{N}^n$, there is a constant $C_{\alpha,\beta}$

$$p_{\alpha,\beta}(a_{\kappa_n}(x,D)) \leq C_{\alpha,\beta} \tag{2.40}$$

which is uniform on $\{\kappa_n\}$.

Each $a_\kappa(x,\eta)$ is shown to be an oscillatory integral with the phase function

$$f_{x,\eta}(y,\xi) = \langle x-y, \xi\rangle - \langle\kappa(x) - \kappa(y), \eta\rangle \tag{2.41}$$

with κ converging to the identity. Before taking sup-norm over x and η to get the semi-norms, the integrals

$$(1+|\eta|)^{-m+|\alpha|}D_\eta^\alpha D_x^\beta a_\kappa(x,\eta) \tag{2.42}$$

are also absolutely bounded oscillatory integrals. The phase functions and amplitude functions for these integrals converge at every point y, η as $\kappa \to id$. Therefore these integrals are continuous with respect to κ in G. $\square$

The continuity of G action on the space Op^m by $g(P)$ is proved in [6].

2.3.4 The distributional kernel

Proposition 2.3.10. *The map*

$$\Psi^m(M;E,F) \to \mathcal{L}(\Gamma_c^\infty(M,E), \Gamma^\infty(M,F)),$$

from a symbol space to its action, is continuous and extends to a bounded linear operator between Sobolev spaces of the sections

$$P: H_s(M,E) \to H_{s-m}(M,F), \tag{2.43}$$

for any real number s. Moreover, the following natural map is continuous:

$$\Psi^m(M;E,F) \to \mathcal{L}_S(H_s(M,E), H_{s-m}(M,F)) \tag{2.44}$$

where in the right hand side ($\mathcal{L}_S$ term) is given the strong topology.

The above proposition follows from its local version, see [22] for the proof.

We now recall the topology on Op^m in [6].

Definition 2.3.11. Let $Op^m = Op^m(M; E, F)$ is the space of all continuous linear maps in

$$\mathcal{L}(\Gamma_c^\infty(M, E), \Gamma^\infty(M, F))$$

which extend to

$$\mathcal{L}_S(H_s(M, E), H_{s-m}(M, F))$$

for all s. Op^m is given the induced limit topology.

Op^m is a Fréchet space. Operators in Op^m do not necessarily have the pseudo-local property, so they contain more than pseudo-differential operators. By definition the natural inclusion map

$$\Psi^m(M; E, F) \to Op^m(M; E, F)) \tag{2.45}$$

is continuous.

Proposition 2.3.12. *The composition map*

$$\Psi^{m_1}(M; E_1, E_2) \times \Psi^{m_2}(M; E_2, E_3) \to \Psi^{m_1+m_2}(M; E_1, E_3) \tag{2.46}$$

is jointly continuous.

Proof. By the closed graph theorem for Fréchet spaces we need only to show separate continuity. Without loss of generality, we assume all three bundles E_i are the same, say E. Let $P_j \to 0 \in \Psi^m(M; E, E)$ and $P \in \Psi^{m'}(M; E, E)$, for any $u \in C_c^\infty(M, E)$, we have $P_j \circ Pu \to 0$ and $P \circ P_j u \to 0$, which imply separate continuity. $\square$

2.3.5 Wave front sets of pseudo-differential operators

In particular, a pseudo-differential operator P has a distributional kernel K_P.

Theorem 2.3.13. *Let $P \in \Psi^m(M; E, F)$, then*

$$WF'(P) \subset \{(x, x, \xi, \xi) : x \in M, \xi \in T_x^*M, \xi \neq 0\}. \tag{2.47}$$

In other words, let K_P be the distributional kernel of P, then

$$WF(K_P) \subset \{(x, x, \xi, -\xi) : x \in M, \xi \in T_x^*M, \xi \neq 0\}. \tag{2.48}$$

Theorem 2.3.14. *For $P \in \Psi^m(M; E, F)$, $u \in \Gamma^{-\infty}(E)$,*

$$WF(Pu) \subset WF(u) \tag{2.49}$$

This property is called the micro-local, or strong pseudo-local property. Yet it can be improved if the symbol of P is smoothing in some directions.

Definition 2.3.15. Let $P : \Gamma_c^\infty(E) \to \Gamma^\infty(F)$ be a pseudo-differential operator. The essential support $\mathrm{Ess}(P)$ of P is the compliment in $T^*M\backslash\{0\}$ of the largest open conic subset of the cotangent bundle on which the symbol has order $-\infty$.

In particular, P is a smoothing operator if and only if $\mathrm{Ess}(P) = \varnothing$.

Proposition 2.3.16. *If $P \in \Psi^m(M; E, F)$ and Γ is a closed conic subset of $T^*M\backslash\{0\}$, the following are equivalent:*

1. *P is of order $-\infty$ in $T * M\backslash\{0\}\backslash\Gamma$;*
2. *$WF'(K_P) \subset \{(\xi, \xi), \xi \in \Gamma\}$;*
3. *for all $u \in \Gamma^{-\infty}(E)$, $WF(Pu) \subset \Gamma \cap WF(u)$.*

So the conclusion in theorem 2.3.14 can be improved to:

$$WF(Pu) \subset WF(u) \cap \mathrm{Ess}(P). \tag{2.50}$$

Definition 2.3.17. When $E = F$, the characteristic set $char(P)$ of P, is the subset of T^*M where the principal symbol is not invertible as bundle morphism.

The following is a generalization of the regularity theorem for the elliptic operators.

Theorem 2.3.18. *(Regularity) For $P \in \Psi^m(M; E, F)$, $u \in \Gamma^{-\infty}(E)$,*

$$WF(u) \subset WF(Pu) \cup char(P). \tag{2.51}$$

In particular when P is elliptic, $char(P) = 0_{T^*M}$ (but the essential support of P is the maximal, $T^*M\backslash\{0\}$); so combining the previous two theorems we have $WF(Pu) = WF(u)$, the elliptic regularity property.

2.3.6 Classical polyhomogeneous symbols

A symbol $p \in S^m(\mathbb{R}^n)$ is called classical polyhomogeneous if p has an asymptotic expansion

$$p(x,\xi) \sim \sum_{j\in\mathbb{N}} p_j(x,\xi) \tag{2.52}$$

where $p_j(x,\xi)$ is homogeneous of degree $m-j$ in ξ for $|\xi|] \geq 1$, that is, if $|\xi| \geq 1$,

$$p_j(x,\xi) = |\xi|^{m-j} p_j(x, \xi/\,|\xi|). \tag{2.53}$$

It is straightforward to extend the definition of classical polyhomogeneous symbols to manifold and bundles, when we may choose a Riemannian metric to define $|\xi_x|$, however, the set of classical polyhomogeneous symbols does not depend on the choice of the metric.

The pseudo-differential operators with classical polyhomogeneous symbols form a closed subalgebra of Ψ^∞, which is also closed under pull-back by diffeomorphism. This subalgebra is the one we are interested, and we shall denote those classical homogeneous subspaces by

$$\mathit{\Psi}^\infty(M;E,F) = \Psi^\infty_{phg}(M;E,F). \tag{2.54}$$

3 Group action and pseudo-differential calculus

3.1 Transversal ellipticity

Definition 3.1.1. For $X \in \mathfrak{g}$, let X_M denote the *fundamental vector field* generated by the action of the one parameter group corresponding to X:

$$X_M f(x) = \left.\frac{d}{dt}\right|_{t=0} f\left(e^{-tX}x\right) \tag{3.1}$$

for any smooth function f on M and $x \in M$.

The G-action decides a map $X \mapsto X_M$ from $\mathfrak{g}$ to $\Gamma(TM)$.

Similarly, for a G-equivariant bundle E over M, the bundle version is a first order differential operator determined by the flow of action generated by X:

$$(X_E s)(x) = \left.\frac{d}{dt}\right|_{t=0} \left(\rho(e^{-tX})s\right)\left(e^{-tX}x\right). \tag{3.2}$$

Note that the $*$-operation on $\Psi^m(E,F)$ is preserved when the action of G preserves the volume form.

Definition 3.1.2. Let

$$\mu_*^{M,G} : T^*M \to \mathfrak{g}^* \tag{3.3}$$

be the adjoint of the group action map defined as: for any $(x,\xi) \in T^*M$

$$\left\langle \mu_*^{M,G}(x,\xi), X \right\rangle = \langle \xi, X_M(x) \rangle . \tag{3.4}$$

The kernel for the moment map is usually denoted by

$$T_G^*M = (\mu_*^{M,G})^{-1}(0) = \{(x,\xi) \in T^*M | \langle \xi, X_M \rangle = 0 \, \forall X \in \mathfrak{g}\}. \tag{3.5}$$

T_G^*M is a G-invariant conic closed subspace of $T^*M \backslash \{0\}$.

Definition 3.1.3. Let $P \in \Psi^k(E,F)$ be a G-invariant pseudo-differential operator. P is called *transversally elliptic* relative to the action of G, if its principal symbol, well defined as an element

$$\sigma_P : S^m(T^*M; Hom(E,F))/S^{m-1}(T^*M; Hom(E,F)) \tag{3.6}$$

is invertible for $(x,\xi) \in T^*M\backslash\{0\}$ as an element in $Hom(E,F)$.

P is called strongly transversally elliptic if there exists a conic neighborhood U of $T_G^*M\backslash\{0\}$ in $T^*M\backslash\{0\}$ and an inverse principal symbol

$$\sigma_Q \in S^{-m}(U; Hom(F,E))/S^{-m-1}(U; Hom(F,E)). \tag{3.7}$$

The strong transversal ellipticity condition is automatic for transversally elliptic classical polyhomogeneous symbols, which is our primary concern.

3.2 Crossed product algebra and action groupoid

Definition 3.2.1. $\mathcal{A} = C_c^\infty(M) \rtimes G$ is defined to be the $*$-algebra of smooth crossed product. That is, $\phi \in \mathcal{A}$ means $\phi \in C_c^\infty(M \times G)$ with product and adjoint:

$$\phi * \psi(x,g) = \int_G \phi(x,h)\psi(h^{-1}x, h^{-1}g) d\mu(h) \tag{3.8}$$

$$\phi^*(x,g) = \overline{\phi(g^{-1}x, g^{-1})}. \tag{3.9}$$

To show that the above product forms a $*$-algebra, the only nontrivial part is the associativity of the product, which can be computed directly. Instead we will only show that this binary operation is a special case of the convolution product of the action groupoid. This fact also supports our similar arguments later on.

A concise definition of a groupoid $\mathcal{G}$ is that a groupoid is a small category (in which all morphism form a set) in which every morphism is invertible. A more detailed definition (such as in [29]) is

$$\mathcal{G} = \left(\mathcal{G}_1, \mathcal{G}_0; \tau, \sigma, \iota, \cdot, ()^{-1}\right) \tag{3.10}$$

with the five maps satisfying the axioms for a small category. We elaborate only on our main example.

Example 3.2.2. The action groupoid is a groupoid $\mathcal{G}_\rho$, where the set of unities $\mathcal{G}_0$ (or objects) is M, and the set of arrows $\mathcal{G}_1$ (or morphism) is $M \times G$, and with the five maps as follows:

(1) The target map $\tau : M \times G \to M$: $\tau(x, g) = x$.

(2) The source map $\sigma : M \times G \to M$: $\sigma(x, g) = \rho(g^{-1})x = g^{-1}x$ (we will sometimes omit the action ρ if no confusion will be caused).

(3) The unity map $\iota : M \to M \times G$: $\iota(x) = (x, e)$.

(4) The partially defined multiplication defined on the subset of $\mathcal{G}_1 \times \mathcal{G}_1$ where the source of the first component matches the target of the second component, is given simply by the multiplication of the group G. That is, $(y, h) \cdot (x, g)$ is defined only when $y = hx$ and

$$(hx, h) \cdot (x, g) = (hx, hg). \tag{3.11}$$

(5) The inverse map $()^{-1} : M \times G \to M \times G$: $(x, g)^{-1} = (g^{-1}x, g^{-1})$.

Remark: our definition of the action groupoid is slightly different from another commonly used in the literature, where typically an arrow (x, g) has source x and target gx. The map $(x, g) \mapsto (g^{-1}x, g)$ gives a groupoid isomorphism between them.

Definition 3.2.3. A groupoid $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_0; \tau, \sigma, \iota, \cdot, ()^{-1})$ is a topological groupoid if all the following are true (1) $\mathcal{G}_1$, $\mathcal{G}_0$ are topological spaces; (2) all five maps are continuous; (3) τ, σ are open maps; (4) ι is a homeomorphism onto its image. A topological groupoid is called locally compact if $\mathcal{G}_1$ is locally compact (so is $\mathcal{G}_0$ as a consequence). $\mathcal{G}$ is called a smooth groupoid when all of the following are true: (1) $\mathcal{G}_1$, $\mathcal{G}_0$ are both smooth manifolds; (2) all five maps are smooth; (3) τ, σ are both submersions; (4) ι is an embedding.

The action groupoid $\mathcal{G}$ is smooth.

For a locally compact topological groupoid, we recall the definition of a Haar measure which gives a convolution algebra.

Definition 3.2.4. A smooth Haar system on a smooth groupoid $\mathcal{G}$ is a family of positive Radon measures $\lambda^{\bullet} = \{\lambda^x : x \in \mathcal{G}_0\}$ on $\mathcal{G}_1$ satisfying the following conditions:

(1) For any $x \in \mathcal{G}_0$, the support of λ^x is in $\mathcal{G}^x = \{\alpha \in \mathcal{G}_1 : \tau(\alpha) = x\}$ and λ^x is a smooth measure on G^x.

(2) (Left invariance) For any $x \in \mathcal{G}_0$ and any continuous function $f : \mathcal{G}^x \to \mathbb{C}$ and any $\alpha \in \mathcal{G}^x$ we have

$$\int_{\mathcal{G}^x} f(\beta) d\lambda^x(\beta) = \int_{\mathcal{G}^{\sigma(\alpha)}} f(\alpha\beta) d\lambda^{\sigma(\alpha)}(\beta) \tag{3.12}$$

(in other words, $\alpha_*(\lambda^{\sigma(\alpha)}) = \lambda^{\tau(\alpha)}$).

(3) (Smoothness) For any $\phi \in C_c^\infty(\mathcal{G})$, the map

$$x \mapsto \int_{\mathcal{G}^x} \phi(\beta) d\lambda^x(\beta) \tag{3.13}$$

is a smooth function on $\mathcal{G}_0$.

Proposition 3.2.5. *A smooth Haar system on a smooth groupoid $\mathcal{G}$ defines a convolution product on $\mathbb{C}_c^\infty(\mathcal{G})$ by*

$$(\phi * \psi)(\alpha) = \int_{\mathcal{G}^{\tau(\alpha)}} \phi(\beta)\psi(\beta^{-1}\alpha) d\lambda^{\tau(\alpha)}. \tag{3.14}$$

And it is a $*$-algebra with

$$\phi^*(\alpha) = \overline{\phi(\alpha^{-1})}. \tag{3.15}$$

For an action groupoid $\mathcal{G}_\rho$, let λ be the Haar measure on G, $\mathcal{G}^x = \{(x, g) : g \in G\}$ is homeomorphic to G by the projection π on the second component. So we simply define $\lambda^x = \pi^*\lambda$. In other words, x is nothing but a label for the copy of G. Left invariance of $\{\lambda^x : x \in M\}$ simply follows from that of G. The rest of the properties in the Haar system definition are also easy to check.

In proposition 3.2.5 let $\alpha = (x, g)$, $\beta = (x, h)$ (since we need $\beta \in \mathcal{G}^{\tau(\alpha)}$), we have $\beta^{-1}\alpha = (h^{-1}x, h^{-1}g)$. It is clear that definition 3.2.1 of the product is just a special case of definition 3.2.5 of the convolution product.

Lemma 3.2.6. *Let E be a G-bundle over M. Denote by $\rho(g) : E_x \to E_{gx}$ the action of an element $g \in G$. Then $\Gamma(M, E)$ is a $\mathcal{A}$-module, with the actions $\phi \in \mathcal{A}$ denoted by ρ, for any $s \in \Gamma(M, E)$,*

$$(\rho(\phi)s)(x) = \int_G \phi(x, g)\rho(g)(s(g^{-1}x))d\mu(g). \tag{3.16}$$

For the proof, instead of direct verification, we relate it to some standard results of groupoid action in the following discussion.

Definition 3.2.7. Let $\mathcal{G}$ be a smooth groupoid, a left $\mathcal{G}$ action a smooth manifold M is a pair (ρ, t), where $t : M \to \mathcal{G}_0$ is a smooth map and

$$\rho : \mathcal{G}_1 \times_{\mathcal{G}_0} M \to M \tag{3.17}$$

where

$$\mathcal{G}_1 \times_{\mathcal{G}_0} M = \{(\alpha, x) \in \mathcal{G}_1 \times M : \sigma(\alpha) = t(x)\}, \tag{3.18}$$

with the following properties

(1) for any $z \in \mathcal{G}_0$, $\iota(z) = t(x)$, we have $\rho(\iota(z), x) = x$,

(2) for all α, β with $\sigma(\alpha) = \tau(\beta)$ and $\sigma(\beta) = t(x)$, we have

$$\rho(\alpha\beta, x) = \rho(\alpha, \rho(\beta, x)). \tag{3.19}$$

When ρ is clear under context, we often use the abbreviations $\rho(\alpha)x$, or even αx for $\rho(\alpha, x)$.

Let E be a hermitian bundle over M and let ρ be a unitary G action on E, which means for each $x \in M$,

$$\rho(g) : E_x \to E_{gx} \tag{3.20}$$

is a unitary.

ρ introduces a $\mathcal{G}_\rho$ action on $U(E)$ (the bundle of fiber-wise unitary transformations on E) in the following way. The groupoid action is given the pair $(\tilde{\rho}, t)$ where $t : U(E) \to M$ is the projection along the fiber, and $\tilde{\rho}$ is defined as

$$\tilde{\rho}((gx, g), u_x) = \rho(g)u_x \in U(E_{gx}) \tag{3.21}$$

for $u_x \in U(E_x)$.

Proposition 3.2.8. *Suppose $\mathcal{G}$ acts on M from the left, $C_c^\infty(M)$ is a left $C_c^\infty(\mathcal{G})$-module by the following action: for $\phi \in C_c^\infty(\mathcal{G})$ and $f \in C_c^\infty(M)$,*

$$(\phi f)(x) = \int_{\mathcal{G}^x} \phi(\alpha) f(\alpha^{-1}x) d\lambda^x(\alpha). \tag{3.22}$$

See [29] for a proof.

From the natural action of $C_c^\infty(U(E))$ on $\Gamma(M, E)$, and combined with the action of $\mathcal{A}$ on $C_c^\infty(U(E))$, the action in lemma 3.2.6 is shown to be the composition.

Lemma 3.2.9. *Let E be a G-vector bundle, for any $\phi \in \mathcal{A}$, the linear continuous operator*

$$\rho(\phi) : \Gamma^\infty(E) \to \Gamma^\infty(E) \tag{3.23}$$

extends to a bounded linear operator on any Sobolev space, that is, for any $s \in \mathbb{R}$,

$$\rho(\phi) : H_s(E) \to H_s(E) \tag{3.24}$$

is bounded.

The proof of the lemma is straightforward.

3.3 Crossed product with pseudo-differential operators

For a manifold with a group action, we will often use operators which are not in the algebra of pseudo-differential operators. For example, in our purpose we frequently use the commutator of a pseudo-differential operator and the action of $\rho(\phi)$, which is in general not pseudo-local. We will define a large enough algebra $\Psi^k(E, E) \rtimes G$ which is, in short, the one generated by both the group action and pseudo-differential operators. $\Psi^k(E, E) \rtimes G$ may also be viewed as groupoid algebra as introduced in the previous section. Here the main advantage of the groupoid viewpoint is that many properties of the new algebra can be easily reproduced.

Similar to definition 3.2.1, we introduce:

Definition 3.3.1. Let E be an hermitian vector bundle over M. We define $\Psi^\infty(E, E) \rtimes G$ as the algebra of families of pseudo-differential operators

$P(g) \in \Psi^k(E,E)$, and the product of $P(g)$ with $Q(g) \in \Psi^l(E,E)$ is a family of pseudo-differential operators in $\Psi^{k+l}(E,E)$,

$$(P * Q)(g) = \int_G P(h) \cdot \left[((h^{-1})_* Q)(h^{-1}g)\right] d\mu(h), \tag{3.25}$$

$$P^*(g) = \overline{g^{-1}P(g^{-1})^* g}. \tag{3.26}$$

Since the composition, the conjugation with $g, h \in G$, and the integration on the parameter space are closed on $\Psi^{k+l}(E,E)$, and the continuity of the integrand over G ensure that the integral (as a Riemann integral over a Lie group with value in a Fréchet space) is still in $\Psi^k(E,E) \rtimes G$. We may show $\Psi^\infty(E,E)$ and $\Psi^0(E,E)$ are algebras by the groupoid argument over a Fréchet algebra.

It follows directly that $\mathcal{A} \subset \Psi^0(E,E) \rtimes G$; $\Psi^0(E,E) \rtimes G$ is not a unital algebra; and $\Psi^\infty(E,E) \rtimes G$ is a $\Psi^\infty(E,E)$-bimodule.

In fact, we are only concerned with the following representation of the action of $\Psi^k(E,E) \rtimes G$ on smooth sections of E, and $\mathcal{H} = L^2(E)$ or in general, any Sobolev space $H^s(E)$. We use the same notions as in lemma 3.2.6.

Proposition 3.3.2. *There is a natural continuous action of $\Psi^0(E,E) \rtimes G$ on $\Gamma^\infty(E)$ defined as follows. For $P = P(g)$ in $\Psi^0(E,E) \rtimes G$,*

$$(Ps)(x) = \int_G P(g)\rho(g)(s(g^{-1}x))d\mu(g), \tag{3.27}$$

The action is extendable to $H_s(E)$ for any $s \in \mathbb{R}$, so the action gives an element in Op^0.

In fact, for any $s \in \mathbb{R}$, an element of $\Psi^k(E,E) \rtimes G$ extends to $H^s(E)$ to $H^{s+k}(E)$, defining an element in Op^k.

Proof. By Lemma 3.2.9 and the uniform continuity of the family $P = P(g)$ over $g \in G$, and the continuity of integration on G, we may routinely check all the above conclusions, based on the corresponding properties of pseudo-differential operators. ☐

From this point on we use the same notation for an element of $\Psi^k(E,E) \rtimes G$ and its representation in Op^k.

An ordinary pseudo-differential operator Q, viewed as a constant family over G. But the algebra of pseudo-differential operators (as constant families) is not a subalgebra of $\Psi^\infty(E,E) \rtimes G$: the composition of two such operators is given by the *-product, — just as convolution of functions is different from pointwise multiplication.

The following compositions between pseudo-differential operators and elements in $\Psi^\infty(E,E) \rtimes G$ are induced from their actions on $\Gamma^\infty(E)$: for $P = P(g) \in \Psi^k(E,E) \rtimes G$ and $Q \in \Psi^k(E,E)$

$$\begin{aligned} (P * Q)(g) &= P(g) \cdot g^*(Q) \\ (Q * P)(g) &= Q \cdot P(g). \end{aligned} \tag{3.28}$$

It is straightforward to check that $\Psi^\infty(E,E) \rtimes G$ is a graded bimodule of pseudo-differential algebra $\Psi^\infty(E,E)$:

$$\begin{aligned} (\Psi^k(E,E) \rtimes G) \cdot \Psi^l(E,E) &\subset \Psi^{k+l}(E,E) \rtimes G \\ \Psi^l(E,E) \cdot (\Psi^k(E,E) \rtimes G) &\subset \Psi^{k+l}(E,E) \rtimes G. \end{aligned} \tag{3.29}$$

3.4 Wave front sets of crossed product algebras

Lemma 3.4.1. *For any $P \in \Psi^k(E,E) \rtimes G$,*

$$WF'(P) \subset \{(g_*\xi_x, \xi_x) : (x,\xi) \in Ess(P(g)),\ \xi_x \in (T^*_G M)_x\} \tag{3.30}$$

Proof. Recall that under the notation, $\rho : M \times G \to M$ is the action of G on M, ρ^*E is the pull-back bundle of E on $M \times G$. As an operator, P is the composition of

$$As(x,g) = \rho(g)(u(g^{-1}x)), \tag{3.31}$$

the family $P(g)$, and

$$Bu(x) = \int_G u(g,x) d\mu(g). \tag{3.32}$$

We now show that $A = (f_A)_*$ and $B = (f_B)_*$ are both push-forward operators, an embedding and a submersion respectively. $f_A : M \times G \to M \times M \times G$

$$f_A(x,g) = (gx, x, g) \tag{3.33}$$

which is the lower part of the bundle map

$$\tilde{f}_A : \rho^*E \to E \boxtimes \rho^*E \tag{3.34}$$

$$\tilde{f}_A : e_{(x,g)} \mapsto \rho(g) e_{gx}; \tag{3.35}$$

$f_B : \rho^* E \to E$ is the projection to the first component, which is the lower part of the bundle map

$$\tilde{f}_B : E \boxtimes \rho^* E \to E \tag{3.36}$$

$$\tilde{f}_B : (e_y, e'_{x,g}) \mapsto e_y. \tag{3.37}$$

It is easy to check that $A = (f_A)_*$ and $B = (f_B)_*$.

By theorem 2.2.12 we have

$$WF'(A) \subset \{(g_*\xi_x, \xi_x, \gamma_g) : \mu_*(\xi_x) = \gamma_g, \} \tag{3.38}$$

where $\mu_* = \mu_*^{M,G}$ is the moment map (with kernel $\{\xi_x \in (T^*_G M)_x\}$) defined earlier. In particular both $WF'_M(A)$ and $WF'_{M\times G}(A)$ are empty. And for the wave relation of $P = P(g)$ we apply theorem 2.3.14, which has empty projections. At last by theorem 2.2.12 we have

$$WF'(B) \subset \mathcal{N}^*_{M\times M}(M \times M \times G) = \{(\eta_y, \xi_x, 0)\}. \tag{3.39}$$

Although it has nonempty right projections, the intersection with the left projection of P is empty. Apply theorem 2.2.10 twice we will reach the conclusion. $\square$

Corollary 3.4.2. *For any $\phi \in \mathcal{A}$,*

$$WF'(\rho(\phi)) \subset \{(g_*\xi_x, \xi_x) :\ (x, g) \in supp(\phi),\ \xi_x \in (T^*_G M)_x\}.$$

3.5 Transversally smoothing operators

Definition 3.5.1. Let $\mathcal{K}_G \subset \Psi^\infty(E, E)$ be the set of pseudo-differential operators that annihilate $\Psi^\infty(E, E) \rtimes G$ modulo $\mathcal{K}$, the ideal of smoothing operators, with composition product. We shall call elements of $\mathcal{K}_G$ transversally smoothing operators.

$\mathcal{K}_G$ can be described in terms of conditions on symbols. It contains those pseudo-differential operators with asymptotically zero symbol on $T^*_G M$. In other words, for $P \in \Psi^\infty$, if

$$\mathrm{Ess}(P) \cap T^*_G(M) = \varnothing \tag{3.40}$$

then $P \in \mathcal{K}_G$.

So $\mathcal{K}_G$ is an ideal if Ψ^∞, containing $\mathcal{K}$ and some pseudo-differential operators of arbitrarily high order.

Example 3.5.2. Fix a pair of conic neighborhood U_1 and U_2 of T_G^*, such that $U_1/\mathbb{R}_+$ is relatively compact in $U_2/\mathbb{R}_+$. There exists a real valued, positive, smooth function χ on $T^*M\backslash\{0\}$ satisfying the following properties: (1) $\chi(\xi) = \chi(\xi/|\xi|))$ for $|\xi| \geq 1$, that is, $\chi(\xi)$ as a symbol is homogeneous of degree 0 for $|\xi| \geq 1$; (2) $\chi(\xi) = 0$ if $\xi \in U_1$ and $|\xi| \geq 1$; (3) $\chi(\xi) = 1$ if $\xi \notin U_2$; (4) $\chi(\xi) = 1$ if $|\xi| \leq 1/2$; (5) χ is G invariant.

Let P_χ be a scalar pseudo-differential operator with symbol χ. Since $\chi = \sigma(P_\chi)$ vanishes on a conic neighborhood of T_G^*M, $P_\chi \in \mathcal{K}_G$.

More transversally smoothing operators can be constructed through this example.

Example 3.5.3. Let $P \in \Psi^m(M; E, F)$ be any pseudo-differential operator of order m. Then by the symbol expansion formula $P_\chi P \in \mathcal{K}_g$ and it generically has order m. Same is true for PP_χ and $P_\chi PP_\chi$.

3.6 Transversal parametrix

Proposition 3.6.1. *Let $P \in \Psi^m(M; E, F)$ be transversally elliptic. Then there exists a $Q \in \Psi^{-m}(M; F, E)$ with principal symbol $\sigma(Q) = \sigma(P)^{-1}$ on T_G^*M, such that $K = 1_F - PQ$ and $K' = 1_E - QP$ are both transversally smoothing.*

For brevity we will call such a Q a *transversal parametrix* for P.

Proof. The construction of the transversal parametrix repeats essentially the construction of the parametrix of an elliptic pseudo-differential operator(see [31]).

In the scope of this proof, we use the sign "$\equiv_V$" for the asymptotic equivalence (up to a difference of degree $-\infty$) on a conic neighborhood V of $T_G^*M\backslash\{0\}$. We omit V when it is $T^*M\backslash\{0\}$.

Let $\sigma(P)$ be the symbol of P, with order m and let Q_0 be a pseudo-differential operator of order $-m$ and with inverse symbol on a conic neighborhood V of T_G^*M. Then $R_0 = Q_0P - 1$ is a pseudo-differential operator, in general not necessarily of negative order, but it has negative ordered symbol on a conic neighborhood of T_G^*M. By composition with an operator of the

form $P_{1-\chi}$ as in example 3.5.2 with $U_1 \subset U_2 \subset V$, we get a symbol for a pseudo-differential operator R of negative order, and

$$\sigma(R) \equiv_{U_1} \sigma(R_0). \tag{3.41}$$

Let C be a pseudo-differential operator with the following asymptotic expansion of its symbol:

$$\sigma(C) \sim \sum_{i=0}^{\infty} (-1)^i \sigma(R^i) \tag{3.42}$$

where $\sigma(R^i)$ is the symbol of the *i-th* power R^i of R. Thus

$$\sigma(C(1+R)) \equiv \sigma(1). \tag{3.43}$$

We claim that $Q = CQ_0$ is a pseudo-differential with properties we need. First we check the properties about $1 - QP$. We have

$$\sigma(QP) \equiv \sigma(C(1+R_0)) \equiv \sigma(C(1+R)) \equiv_{U_1} \sigma(1). \tag{3.44}$$

To prove the properties about $1_F - PQ$, we may construct Q' a similarly so $1 - PQ' \equiv_{U_1} 0$. But with the existence of such a Q' we know Q work in place of Q':

$$\sigma(Q) - \sigma(Q') \equiv_{U_1} \sigma(QPQ') - \sigma(QPQ') = 0, \tag{3.45}$$

therefore

$$\sigma(1 - PQ) \equiv_{U_1} \sigma(1 - PQ') + \sigma(P(Q' - Q)) \equiv_{U_1} 0. \tag{3.46}$$

□

Consequently, $1 - PQ$ and $1 - QP$ composed with any operator in Ψ_G^∞ is smoothing. In particular, their composition with any $\rho(\phi)$ for $\phi \in \mathcal{A}$ is smoothing.

3.7 Summary of algebraic properties

We summarize some facts about the algebra $\Psi^k(E,E) \rtimes G$ and the pseudo-differential algebra $\Psi^k(E,E)$.

1. $\Psi_G^k \subset \Psi_G^l$ for $k \leq l$;

2. $\Psi_G^k \cdot \Psi_G^l \subset \Psi_G^{k+l}$;
3. $\Psi^k \cdot \Psi_G^l \subset \Psi_G^{k+l}$; $\quad \Psi_G^k \cdot \Psi^l \subset \Psi_G^{k+l}$;
4. $\Psi_G^0 \subset B(\mathcal{H})$;
5. When $r > 0$, $A \in \Psi_G^{-r}$ is compact;
6. $\Psi_G^k \cdot \mathcal{K}_G \subset \mathcal{K}$; $\quad \mathcal{K}_G \cdot \Psi_G^k \subset \mathcal{K}$;
7. $\Psi_G^{-\infty} \subset \mathcal{K}$.

4 Trace formula and spectral analysis

4.1 Trace formula on Ψ_G

For $\mathcal{H} = L^2(E)$, we recall that $\mathcal{L}^1(\mathcal{H})$ is the trace class operators and for $A \in \mathcal{L}^1(\mathcal{H})$, $\|A\|_1$ is the trace class trace class norm.

Proposition 4.1.1. *If $k < -\dim M - 1$, then the extension from smooth sections to L^2 sections sends*

$$\Psi^k(E, E) \rtimes G \subset \mathcal{L}^1(\mathcal{H}). \tag{4.1}$$

Moreover the above embedding from the the Fréchet algebra $\Psi^k(E, E) \rtimes G$ to the Banach space $(\mathcal{L}^1(\mathcal{H}), \|\cdot\|_1)$ is continuous.

Proof. We take a self-adjoint, elliptic, invertible pseudo-differential operator, such as the second order invariant Laplacian $1 + \Delta$ on scalar functions $C^\infty(M)$. $A(1 + \Delta)^{-k/2}$ is bounded, and $(1 + \Delta)^k$ is trace class by the Weyl's formula:

$$\mu_n(1 + \Delta) = c_n n^{\dim M/2}(1 + o(1/n)) \tag{4.2}$$

where the positive c_n depends only on n and the volume of M.

To show the trace class norm is bounded on Ψ_G^k, we can use

$$\|A\|_1 \leq \|A(1 + \Delta)^{-k/2}\|_{B(\mathcal{H})}\|(1 + \Delta)^{k/2}\|_1, \tag{4.3}$$

together with joint continuity of the multiplicative map in Ψ. $\square$

The above argument gives a stronger result than the following one by just looking at the distributional kernel. For k sufficiently negative, the distributional kernel of $A \in \Psi_G^k$ is C^r for all $r < -k - \dim M$. Stinespring [36] showed that if an integral operator is trace class if its kernel is C^r with $r \geq [\dim M/2] + 1$. However in proposition 4.1.1 the condition on k can be relaxed in most interesting cases to

$$k < -\dim M_0/G \tag{4.4}$$

where M_0 is the union of all of principal orbits of the G-action, an open dense subset of M. (see also [11]).

Also for $P \in \Psi_G^k$, $k < -\dim M - 1$, we have a well defined Fredholm determinant

$$\det(I + \lambda P) \in \mathbb{C} \tag{4.5}$$

for any $\lambda \in \mathbb{C}$ and is an entire function on $\lambda \in \mathbb{C}$. The Fredholm determinant is a useful tool in the study of trace and eigenvalue related problems. We refer to [17] for details. However we do not use the Fredholm determinant here, except for emphasizing its role in the proof of the Lidskii's trace theorem.

Proposition 4.1.2. *Assume $k < -\dim M - 1$. For $P = \{P(g)\} \in \Psi_G^k$ let*

$$K_P(x, y) \in \Gamma^\infty(E \boxtimes E) \tag{4.6}$$

be the continuous representation of the distributional kernel of $P(g)$. Then

$$Trace(P) = \int_G \int_M tr_x(K_P(x, gx)\rho_x(g))dvol(x)d\mu(g), \tag{4.7}$$

where tr_x is the fiber-wise trace on $E_x \otimes E_x$.

Proof. By the continuity of the trace under the trace class norm, we easily reduce the statement to its version on Ψ^k: for $R \in \Psi^k(E, E)$ with distributional kernel

$$K_R(x, y) \in \Gamma^\infty(E \boxtimes E)$$

which is continuous,

$$Trace(R) = \int_M tr_x(K_P(x, x))dvol(x). \tag{4.8}$$

To prove (4.8), we first prove it for $R \in \Psi^{-\infty}$. There (4.8) holds since the set of operators of finite rank is dense under the topology defined by the

trace class norm. Next we observe the right hand side of (4.8) is a unique continuous extension of the trace from $\Psi^{-\infty}$ to Ψ^k.

By the Lidskii trace theorem [17], any such extension of the trace on operators of finite rank must be equal to

$$Trace(R) = \sum \mu_n(P) \tag{4.9}$$

where $\mu_n(P)$ is the n-th non-zero eigenvalue of P counting multiplicity (for a compact operator the nonzero eigenvalues have finite multiplicity); so it is the unique trace. The harder part in the proof of the Lidskii's theorem is to show that on operators with no eigenvalue, any reasonably continuous extension of the trace vanishes.

□

4.2 Asymptotic analysis of oscillatory integrals

4.2.1 Localization of trace formula

In the trace formula (4.7) it is not convenient to relate the distributional kernels to the symbols of the operators. We try to decompose the global formula into a sum of terms which are all local in coordinate charts.

To do this we first choose a finite atlas of M, on each chart in the atlas the bundles are trivial. We take a refinement of this atlas so that each new chart has a domain which is relative compact subset of the domain of any chart in the old atlas. Then we use a partition of unity on G fine each so that we can assume on each small piece the G action on any of the domains of the new atlas. Next by a partition of unity on M, we can decompose (4.7) into the sum of integrals on open subset $U \times G$ such that $g(U)$ is contained in the same chart for all g in the support of $g \mapsto P(g)$.

Locally the kernel of a pseudo-differential operator $Q(g)$ is of the form

$$K_{Q(g)}(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q(x, \xi, g) d\xi, \tag{4.10}$$

in particular,

$$K_{Q(g)}(x, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} q(x, \xi, g) d\xi. \tag{4.11}$$

But for $Q = \{Q(g)\}$, the kernel on the diagonal is of the form:

$$Q(x, x) = \int_G \int_{\mathbb{R}^n} e^{i(x-gx)\cdot\xi} p(x, \xi, g) d\xi dvol(x) d\mu(g) \tag{4.12}$$

and it is an oscillatory integral with phase function $(x - gx) \cdot \xi$.

In the case there is no group action, equivalently $G = \{e\}$, we find that the phase function

$$\phi(x, g, \xi) = \langle x - gx, \xi \rangle \tag{4.13}$$

in formula (4.12) is zero hence no oscillatory integrals is directly involved in the kernel.

4.2.2 Asymptotic expansion of trace

Here we use a simple example to show the need to study the asymptotic behavior of oscillatory integrals.

We take the scalar functions on $\mathbb{R}^n$. Given a strictly positive, elliptic differential operator P, so that the symbol $p(x, \xi)$ is homogeneous of degree $m > 0$. We shall need to analyze the behavior of the resolvant $(P - \lambda)^{-1}$ for λ on the negative half of the real axis, and let $|\lambda|$ tends to ∞.

If $m > n$, the resolvant is trace class for any $\lambda \in (-\infty, 0)$, since it is a pseudo-differential operator of order $-m$.

$$Trace((P - \lambda)^{-1}) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-gx,\xi)}(p(x, \xi) - \lambda)^{-1}\, d\xi\, dx. \tag{4.14}$$

Suppose we want to study how the trace varies with respect to λ. For convenience, let $\lambda = -\mu^m$ where μ is positive, let t be $|\xi|$, and let $\xi_1 = \xi/t$ be the unit vector.

As in [33] and [34], we treat the symbol

$$(p(x, \xi) - \lambda)^{-1} = (t^m p(x, \xi_1) + \mu^m)^{-1} \tag{4.15}$$

as a homogeneous symbol in (ξ, μ). With a substitution $t = \mu s$ and then rename $\xi = s\xi_1$ the trace integral looks like

$$(2\pi)^{-n}\mu^{-m} \int_{\mathbb{R}^{2n}} e^{i\mu(x-gx,\xi)}(p(x, \xi) + 1)^{-1}\, d\xi\, dx. \tag{4.16}$$

Note that $\mu = (-\lambda)^{1/m}$ appears only outside of the integral and as a linear factor of the phase function. This showcase demonstrates why the asymptotic behavior of oscillatory integral appears. It also shows when G action is trivial there is no oscillatory integral involved.

4.3 Asymptotic expansion of oscillatory integrals

4.3.1 The question

We are interested in integrals of the form (4.12) with $p(x, g, \xi)$ polyhomogeneous in ξ and with an extra parameter μ. The asymptotic behavior when $|\xi|$ and μ are large is a major concern, so we formulate the abstract oscillatory integral as

$$I_{\phi,u}(\tau) = \int_{\mathbb{R}^N} e^{i\tau\phi(y)} u(y) dy \tag{4.17}$$

where $\tau > 0$, u is smooth. In practice, ϕ and u may also contains parameters. The question is to decide the asymptotic behavior of the integral (4.17) as τ tends to ∞.

The derivative of the phase function ϕ' (with respect to y) matters the most in the pattern of the possible asymptotic expansion.

The first basic fact is that ϕ' is away from zero, then in (4.17)

$$I_{\phi,u}(\tau) = o(\tau^{-N}) \tag{4.18}$$

for any $N > 0$.

The next basic fact is that if ϕ' is a Morse function, then we can write explicit asymptotic expansions in τ with coefficients from information about germs of ϕ and u at the critical points of ϕ'. Such asymptotic expansions are the basic in determining the asymptotic behavior of symbols under product, adjoint, change of variables. So they are the basic supporting techniques in pseudo-differential calculus. See [21], chapter 7 for details.

4.3.2 General phase function

The phase function we are interested in has critical points contained in

$$\{(x, \xi, g) \in T_1^*M \times G : gx = x \text{ and } (dg)_x^*\xi = \xi\}. \tag{4.19}$$

which at least contains the subset

$$T_1^*M \times \{e\}. \tag{4.20}$$

In the above T_1^*M means the quotient $(T^*M\backslash\{0\})/\mathbb{R}_+$. So the phase function is never a Morse function.

By recent results of [23] and [28], related to a generalization of Hilbert's seventh problem to group action case, we may assume that the action of G

is analytic on a compatible analytic atlas of M. This atlas exists as shown in [23], and the uniqueness is proven in [28]. The phase function is analytic under this atlas.

Qualitative results were introduced for any analytic phase function, using resolution of singularities originated from Atiyah [3]. The asymptotic expansion formula has been given by many people including Bernstein, Malgrange, etc.. There is an excellent source for of backgrounds, applications, explanations of the result we use here. It is in part II (page 169–268) of volume II of the two volume monographs by Arnold et al [1], [2]. Although it is too long a topic to elaborate on, all we need is one technical theorem by Malgrange.

4.3.3 Malgrange's theorem

In [2], the following theorem is derived and illustrated with examples using Newton polyhedra of the singularities.

Theorem 4.3.1. *Let ϕ be a real valued nonzero analytic function on $\mathbb{R}^N$. For $u \in C_c^\infty(\mathbb{R}^N)$ (real or complex valued) a test function, let $I(\tau)$ be the oscillatory integral*

$$I_{\phi,u}(\tau) = \int_{\mathbb{R}^N} e^{i\tau\phi(x)} u(x) dvol(x). \tag{4.21}$$

Then for $\tau \to \infty$

$$I(\tau) = \sum_{\alpha,p,q} c_{\alpha,p,q}(u) \tau^{\alpha-p} (ln\, \tau)^q, \tag{4.22}$$

where $\alpha \leq 0$ runs through a finite set of rational numbers, $p, q \in \mathbb{N}$ and $0 \leq q < n$. Moreover $c_{\alpha,p,q}$ are all distributions with support inside

$$S_\phi = \{x \in \mathbb{R}^N : d\phi(x) = 0\}, \tag{4.23}$$

and with finite orders not exceeding N.

4.4 Noncommutative residue

4.4.1 Definition

In Connes-Moscovici local index formula, we need to study the trace of AP^{-z}, where $A \in \Psi_G$ and $P \in \Psi$ is a second-order, self-adjoint, positive, elliptic pseudo-differential operator. In fact P is also G-invariant but we do not use

this assumption now. For convenience, P is assumed to be a second order, since we will have any complex power of P and can easily change the scale of z. We choose to work on an elliptic operator instead of a transversally elliptic one because with all the other assumptions we can add to P a part K which is transversally smoothing and make $P+K$ is elliptic (see section 4.6). It will be shown that such a K has negligible contribution to the asymptotic expansions we are interested in.

In this section we discuss the noncommutative residues defined by such a P. In fact, to allow some room for perturbation of P relax the assumption on the spectrum of P. In stead of asking

$$Spec(P) \subset (\epsilon, \infty), \tag{4.24}$$

we allow an open neighborhood of it. This neighborhood can be assumed to be a sector, as large as to miss the negative half of the real axis.

In summary we assume $P \in \Psi(M; E)$ is a second order, elliptic, with

$$Spec(P) \subset \mathbb{C}\backslash\{0\}\backslash\{\text{a conic neighborhood of } \mathbb{R}_-\}. \tag{4.25}$$

Since P is invertible and P^{-1} is compact, $Spec(P)$ is a discrete set.

For λ on a contour along the negative real axis C, the resolvant $(P-\lambda)^{-1}$ exists and is bounded. Let $\ln z$ denote the unique branch of the multiple valued logarithm function defined on $\mathbb{C}\backslash\{0\}\backslash\mathbb{R}_-$ such that the imaginary part of it $arg(z) = 0$ for $z \in R_+$. Then $\ln(-\lambda)$ is defined on $\mathbb{C}\backslash\{0\}\backslash\mathbb{R}_+$ and with imaginary part $arg(\ln(-\lambda)) = 0$ on the negative half of the real axis. Seeley ([33], [20], [34]) showed that all the complex powers P^z of P are pseudo-differential.

The function

$$z \mapsto Trace(AP^{-z}) \tag{4.26}$$

is well defined and analytic on $Re(z) > \dim M$.

Definition 4.4.1. For all $A \in \Psi^\infty(M; E)$, assuming that $Trace(AP^{-z})$ extends meromorphically the whole plane, we define

$$\tau_k^P(A) = Res_{z=0}(z^k Trace(AP^{-z})). \tag{4.27}$$

4.4.2 Equivalent definitions

Lemma 4.4.2. *Suppose N is a fixed positive integer, $A \in \Psi^\infty(M) \rtimes G$, s_p are strictly decreasing sequences in $\mathbb{R}$. Suppose for some fixed positive integer*

K (typically K is chosen to be $(\dim M + ord(A))/2$), $A(P-\lambda)^{-K}$ is trace class. Then the following three statements are equivalent:

(A) For $\lambda \to \infty$ on any ray of $\mathbb{C}\backslash\mathbb{R}_+$,

$$Tr(A(P-\lambda)^{-K}) \sim \sum_{l\geq 0}\left(\sum_{q=0}^{N} b_{p,q}(\ln(-\lambda))^q\right)(-\lambda)^{-K+s_p}. \tag{4.28}$$

(B) When $t \to 0+$

$$Tr(Ae^{-tP}) \sim \sum_{p\geq 0}\left(\sum_{q=0}^{N} b'_{p,q}(\ln t)^q\right)t^{-s_p}. \tag{4.29}$$

(C) The zeta function $\Gamma(z)Tr(AP^{-z})$, which converges for $Re(z) > K$, extends to a meromorphic function on $\mathbb{C}$, with the poles described as follows: up to a holomorphic function,

$$\Gamma(z)Tr(AP^{-z}) \sim \sum_{p\geq 0}\left(\sum_{q=0}^{N}\frac{b''_{p,q}}{(z-s_p)^{q+1}}\right). \tag{4.30}$$

Moreover, for any p, the coefficient subsets $\{b_{p,q} : q = 0,\dots,N\}$, $\{b'_{p,q} : q = 0,\dots,N\}$, $\{b''_{p,q} : q = 0,\dots,N\}$ determine each other.

Proof. The proof is based on the following straightforward integral transformation formulas ([19]):

$$\begin{aligned} P^{-s} =& \frac{1}{(s-1)\cdots(s-k)}\frac{i}{2\pi}\int_{C_1}\lambda^{k-s}\partial_\lambda^k(P-\lambda)^{-1}d\lambda \\ =& \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tP}dt, \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} e^{-tP} =& t^{-k}\frac{i}{2\pi}\int_{C_1}e^{-t\lambda}\partial_\lambda^k(P-\lambda)^{-1}d\lambda \\ =& \frac{1}{2\pi i}\int_{Re(s)=c}t^{-s}\Gamma(s)P^{-s}ds, \end{aligned} \tag{4.32}$$

where $C_r, r \in (0,1]$ is the counterclockwise contour

$$\begin{aligned} C_r =& \{z \in \mathbb{C} : arg(z) = \pm(\pi/2-\delta), |z| \geq r\} \\ & \cup \{z \in \mathbb{C} : arg(z) \in [-\pi/2+\delta, \pi/2-\delta], |z| = r\} = rC_1 \end{aligned} \tag{4.33}$$

for some fixed small $\delta > 0$. Equations (4.31) (4.31) are the Mellin transform, the inverse Mellin transform and contour integrals that can be converted to a slightly tilted version of Mellin transform. Including the appearance of higher logarithmic terms, the argument has been given in [34], see also [25]. Here we only show the relation between the coefficients. Note that the contribution from a different contour only produce a smoothing operator.

To show (A) $\Rightarrow$ (B): we have

$$A(P-\lambda)^{-s} \in \Psi^{-K+ord(A)}(E,E) \rtimes G$$

is trace class by the choice of K. By

$$\partial_\lambda^k (P-\lambda)^{-1} = k!(P-\lambda)^{-k-1}, \tag{4.34}$$

we get

$$Ae^{-tP} = (N-1)!t^{-N+1}\frac{i}{2\pi}\int_{C_1} e^{-t\lambda}A(P-\lambda)^{-N}d\lambda. \tag{4.35}$$

Direct calculation shows

$$t^{-N+1}\frac{i}{2\pi}\int_{C_1} e^{-t\lambda}(-\lambda)^{r_j-N}d\lambda = \frac{t^{-r_j}}{\Gamma(-r_j)} \tag{4.36}$$

and similarly

$$\begin{aligned} &t^{-N+1}\frac{i}{2\pi}\int_{C_1} e^{-t\lambda}(-\lambda)^{s_p-N}(\ln(-\lambda))^p d\lambda \\ &= t^{-s_p}\frac{i}{2\pi}\int_{C_t} e^{-\lambda}(-\lambda)^{s_p}(\ln(-\lambda)-\ln t)^q d\lambda \\ &\sim \sum_{i=0}^{q}(-1)^i\frac{q!}{i!(q-i)!}t^{-s_p}(\ln t)^i\frac{i}{2\pi}\int_{C_1} e^{-\lambda}(-\lambda)^{s_p}(\ln(-\lambda))^{q-i}d\lambda \\ &\sim \sum_{i=0}^{q}(-1)^i\frac{q!}{i!(q-i)!}t^{-s_p}(\ln t)^i(\frac{1}{\Gamma})^{(q-i)}(-s_p), \end{aligned} \tag{4.37}$$

where $(\frac{1}{\Gamma})^{(q-i)}$ is the $(q-i)$-th derivative of the entire function $1/\Gamma$. By the first equality of (4.32), and the fact that in this integral transform asymptotic expansions in $-\lambda$ correspond to asymptotic expansions in t.

To see the relation between coefficient coefficients, we look at the contribution of a single term,

$$b''_{p,q} = \sum_{i=0}^{q}(-1)^i\frac{q!}{i!(q-i)!}t^{-s_p}(N-1)!b_{p,i}(\frac{1}{\Gamma})^{(q-i)}(-s_p). \tag{4.38}$$

The above gives $b''_{p,q}$ recursively.

The rest of details can be found in, for example, [34] and [25]. $\square$

4.5 Asymptotic trace formula for weakly polyhomogeneous parametrized pseudo-differential operators

4.5.1 Weakly parametric pseudo-differential operators

For a self-adjoint, elliptic pseudo-differential operator P of positive order, the analysis of the resolvent $(P-\lambda)^{-1}$ for λ in a certain sector Γ of $\mathbb{C}\backslash\mathbb{R}_+$ plays a very important role in the study of noncommutative residues. In this section we recall some basic definition and results by Grubb and Seeley [20].

Let Γ be a sector of $\mathbb{C}\backslash\mathbb{R}_+$, of the form

$$\{\lambda = re^{i\theta} | r > 0, \theta \in I \subset (0, 2\pi)\}, \tag{4.39}$$

where I is a subinterval, let Γ^o be its image under conjugacy which has the same form. We are mainly interested in the case when Γ is near the negative real axis but some discussion apply to any sector on $\mathbb{C}\backslash\{0\}$.

For simplicity we start with symbols in $M = \mathbb{R}^n$ and with scalar values. What matters for us is the behavior of $r = |\mu| \to \infty$ for $\mu \in \Gamma^o$, or equivalently, the behavior of $z = 1/\mu \to 0$ for $z \in \Gamma$, which we use more often.

Definition 4.5.1. The *weakly parametric* symbol space $S^{m,0}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ consists of functions $p(x, \xi, \mu)$ that are holomorphic in $\mu = 1/z \in \Gamma^o$ for

$$|\xi, \mu| = (|\xi|^2 + |\mu|^2)^{1/2} \geq \epsilon \tag{4.40}$$

for some ϵ and satisfy, for all $j \in \mathbb{N}$, $1/z \in \Gamma$,

$$\partial_z^j p(\cdot, \cdot, 1/z) \in S^{m+j}(\mathbb{R}^n \times \mathbb{R}^n) \tag{4.41}$$

locally uniformly in $|z| \leq 1$. Moreover, for any $d \in \mathbb{C}$, define

$$S^{m,d} = \mu^d S^{m,0}. \tag{4.42}$$

That is, $p(\cdot, \cdot, 1/z) \in S^{m,d}$ if $z^d p(\cdot, \cdot, 1/z)$ satisfies (4.41).

$S^{m,d}$ are Fréchet spaces with semi-norms implied in the definition.
The following properties are immediate:
1. As symbols constant on Γ, $S^m \subset S^{m,0}$.
2. If $m < 0$, then $f(x,\xi) \in S^m$ implies $p(x,\xi,\mu) = f(x,\xi/\mu) \in S^{0,0}$.
3. $S^{m,d} \subset S^{m',d'}$ for $m \leq m'$, $d' - d \in \mathbb{N}$.
4. The following maps are continuous

$$\begin{aligned}
&\partial_\xi^\alpha : S^{m,d} \to S^{m-|\alpha|,d} \\
&\partial_x^\beta : S^{m,d} \to S^{m,d} \\
&z^k : S^{m,d} \to S^{m,d-k} \\
&\partial_z^j : S^{m,0} \to S^{m,0} \\
&\partial_z : S^{m,0} \to S^{m+1,0} + S^{m,d+1} \; d \neq 0
\end{aligned} \tag{4.43}$$

5. $S^{m,d} \cdot S^{m',d'} \subset S^{m+m',d+d'}$.
As usual, let

$$S^{\infty,d} = \bigcup_{m \in \mathbb{R}} S^{m,d}, \quad S^{-\infty,d} = \bigcap_{m \in \mathbb{R}} S^{m,d}. \tag{4.44}$$

Definition 4.5.2. Let p_j, $j \in \mathbb{N}$ be a sequence of symbols in $S^{m_j,d}$, where $m_j \downarrow 0$. Then an asymptotic expansion in $S^{m_0,d}$ (or $S^{\infty,d}$) for $p \in S^{m_0,d}$

$$p \sim \sum_{j \in \mathbb{N}} p_j \tag{4.45}$$

means that for any $N \in \mathbb{N}$,

$$p - \sum_{j=0}^{N} p_j \in S^{m_{N+1},d}. \tag{4.46}$$

The following is a generalization of *classical polyhomogeneous* symbols, which has asymptotic expansion of homogeneous in ξ of integer-degree (see, for example, [34]). Pseudo-differential operators with classical polyhomogeneous symbols are called classical pseudo-differential operators.

Definition 4.5.3. $p \in S^{\infty,d}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ is called *weakly polyhomogeneous* if there exists a sequence of symbols $p_j \in S^{m_j - d,d}$, homogeneous in (ξ, μ) for $|\xi| \geq 1$ of degree $m_j \downarrow -\infty$, such that $p \sim \sum p_j \in S^{\infty,d}$.

Theorem 4.5.4. *For $p \in S^{m,d}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$, the limits*

$$p_{(d,k)}(x,\xi) = \lim_{z\to 0} \partial_z^k (z^d p(x,\xi,1/z)) \in S^{m-k}(\mathbb{R}^n \times \mathbb{R}^n) \tag{4.47}$$

exists and for any $N \in \mathbb{N}$,

$$p(x,\xi,\mu) \sim \sum_{k=0}^{N} \mu^{d-k} p_{(d,k)} \in S^{m+N+1,d-N-1}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma). \tag{4.48}$$

Proposition 4.5.5. *For a parameterized pseudo-differential operator $P(\lambda)$ on $\mathbb{R}^n$ with symbol $p(x,\xi,\lambda) \in S^{-\infty,d}$, its kernel has an expansion*

$$K(x,y,\lambda) \sim \sum_{k\in\mathbb{N}} \lambda^{d-k} K_k(x,y), \tag{4.49}$$

with K_k smooth on $\mathbb{R}^n \times \mathbb{R}^n$, and

$$K - \sum_{k=0}^{N} K_k(x,y)\mu^{d-k} \in C^\infty(\mathbb{R}^{2n} \times \Gamma), \tag{4.50}$$

holomorphic in $\mu \in \Gamma^o$ for $|\mu| \geq 1$.

In particular for smooth functions ϕ and ψ with disjoint support, any $P(\lambda)$ with symbol in $S^{m,d}$, $\phi P(\lambda)\psi$ is such an example.

Definition 4.5.6. A parameterized pseudo-differential operator $P(\mu)$ on M, $\mu \in \Gamma^o$ is in $\Psi^{m,d}(M)\times\Gamma$ if for any $\phi, \psi \in C_c^\infty(M)$, with support in a common coordinate system U, $\phi P(\mu)\psi$ has a symbol in $S^{m,d}(U \times \mathbb{R}^n \times \Gamma)$.

So the noncommutative residues, are basically the coefficients of the pure logarithmic grow and its powers in the asymptotic expansion of the resolvent.

Definition 4.5.7. $\Psi^{-\infty,d}(M) \times \Gamma$, consists of operators parameterized in Γ with kernel $K(x,y,\mu)$, smooth in (x,y), holomorphic in μ for $\mu \in \Gamma^o$, $|\mu| \geq 1$, and has expansion

$$K(x,y,\mu) \sim \sum K_k(x,y,\mu)\mu^{d-k} \tag{4.51}$$

with

$$\partial_x^\alpha \partial_\xi^\beta (K - \sum_{k=0}^{J} K_k \mu^{d-k}) = O(\mu^{d-J-1}) \tag{4.52}$$

for $|\mu| \to \infty$ locally uniformly.

All these concepts are of local natural, so the above symbolic calculus extends to to the vector bundle version. By a partition of unity subordinate to a trivialization of the vector bundles, we define $S^{m,d}(M \times \Gamma; E, F)$ and $\Psi^{m,d}(M \times \Gamma; E, F)$. in the usual way (see [20] for details).

Theorem 4.5.8. *Let $p(x, \xi) \in S^r(T^*M; E)$ be a weakly homogeneous symbol of positive order $m \in \mathbb{N}_+$ (i.e., m-homogeneous for $|\xi| \geq 1$), and assume $(p(x, \xi) + \mu^m)$ invertible on a closed sector Γ, then it is weakly polyhomogeneous on μ, with weakly parameterized symbols*

$$(p(x, \xi) + \mu^m)^{-1} \in S^{-m,0}(\Gamma) \cap S^{0,-m}(\Gamma). \tag{4.53}$$

This theorem shows that Grubb and Seeley generalized Seeley's ([33]) and Shubin's ([34]) methods in spectral analysis for differential operators. For example when a scalar symbol p is strictly positive, there is a sector Γ near negative real axis such that $\mu = (-\lambda)^{1/m}$ has a unique root for $\lambda \in \Gamma$, satisfying the invertibility condition.

4.5.2 Asymptotic expansion of local trace formula

For a family of pseudo-differential operators with parameters (g, μ), uniformly in G, and and weakly parametric in μ in a sector Γ, we find the asymptotics the kernel and trace of an operator in $\Psi^{-n-1,d}(E, E; \Gamma) \rtimes G$.

In particular we are interested in traces of operators of the form

$$P(D - \lambda)^{-k} \tag{4.54}$$

where D is positive, positively ordered, polyhomogeneous, elliptic pseudo-differential operator and for what we concern λ is on the negative half of the real axis.

As explained in the introductory part of section 4.2, we may work on a relatively compact neighborhood of $\mathbb{R}^n$. By proposition 4.5.5, the contribution form a smoothing operator is already known, we only consider operators modulo a smoothing operator.

Also the scalar case is sufficient, in the matrix case we take tr_x and check the invariance under transition functions.

In $\mathbb{R}^n$, a pseudo-differential operator can be written as the Fourier integral form

$$Op(p)u(x) = \int \int e^{i(x-y)\cdot\xi} p(y, \xi) u(y) dy d\xi, \tag{4.55}$$

and the distributional kernel is the integral on $\mathbb{R}^n(\xi)$, which has possible singularities only on the diagonal.

Proposition 4.5.9. *Let $d \geq 0$, $U \subset \mathbb{R}^n$ be a relatively compact open chart invariant under G action,*

$$p(x,\xi,g,\mu) \sim \sum_{j=0}^{\infty} p_j(x,\xi,g,\mu) \in S^{-n+d,d}(U \times \mathbb{R}^n, \Gamma, G) \tag{4.56}$$

be a weakly polyhomogeneous in (ξ,μ), uniformly in G, with p_j homogeneous with degrees $m_j \downarrow -\infty$. Then $\int_G Op(p)\rho(g)dg$ has a continuous kernel $K_p(x,y,\mu)$ with a diagonal expansion

$$K_p(x,x,\mu) \sim \sum_{j=0}^{\infty} c_j(x)\mu^{m_j+n} + \sum_{\alpha,p,q} c'_{\alpha,p,q}(p)(x)\mu^{d+\alpha-p}(\ln\mu)^q \tag{4.57}$$

for $|\mu| \to \infty$, locally uniformly in Γ, and for α, p, q described in Theorem 4.3.1. All terms are decided by the symbol expect $c'_{\alpha,p,0}$. The contribution to each c_j, $c'_{\alpha,p,q}$ are from finitely many p_j's.

The following proof is a adapted from the proof of theorem 2.1 in Grubb-Seeley [20].

Proof. Without loss of generality we may assume $d = 0$, the general case follows by considering $\mu^d p$.

Let r_J be remainder terms:

$$r_J = \sum_{j=0}^{J} p_j. \tag{4.58}$$

$\int_G Op(r_J)\rho(g)\,dg$ has kernel

$$K_{r_J}(x,x,\mu) = \int_{\mathbb{R}^n}\int_G e^{i(x-gx)\cdot\xi} r_J(x,\xi,\mu)dg\,d\xi \tag{4.59}$$

so by theorem 4.5.4, for any $N \in \mathbb{N}_+$

$$r_J(x,\xi,\mu) + \sum_{k=0}^{N} s_k(x,\xi)\mu^{-k} + O(|\xi|^{m_j-N}\mu^{-N}), \tag{4.60}$$

where $s_k \in S^{m_j+k}$. For any N, taking large enough J will yield

$$K_{r_J}(x,x,\mu) = \sum_{k=0}^{N} c_{N,J,k}(x)\mu^{-k} + O(\mu^{-N}) \tag{4.61}$$

so this will contribute like smoothing operators.

Now we discuss the contribution of each homogeneous symbols $p_j \in S^{m_j,0}$ which is homogeneous in (ξ,μ) of degree m_j. As usual, we split the integral into three parts:

$$\begin{aligned} K_{p_j}(x,x,\mu) &= \int_{\mathbb{R}^n}\int_G e^{i(x-gx)\cdot\xi} p_j(x,\xi,\mu)dg\, d\xi \\ &= \int_{|\xi|\geq|\mu|}\int_G e^{i(x-gx)\cdot\xi} p_j(x,\xi,\mu)dg\, d\xi \\ &+ \int_{|\xi|\leq 1}\int_G e^{i(x-gx)\cdot\xi} p_j(x,\xi,\mu)dg\, d\xi \\ &+ \int_{1\leq|\xi|\leq|\mu|}\int_G e^{i(x-gx)\cdot\xi} p_j(x,\xi,\mu)dg\, d\xi. \end{aligned} \tag{4.62}$$

For the second and third integral, we use theorem 4.5.4 again,

$$p_j(x,\xi,\mu) = \sum_{k=0}^{M} \mu^{-k} q_k(x,\xi) + R_M(x,\xi,\mu), \tag{4.63}$$

where

$$q_k(x,\xi) = \frac{1}{k!}\partial_z^k p(x,\xi,1/z) \in S^{m_j+k} \tag{4.64}$$

and homogeneous in ξ for $|\xi| \geq 1$ of degree $m_j + k$, and

$$R_M = O(|\xi|^{m_j+M}\,\mu^{-M}). \tag{4.65}$$

The second integral only contribute to non-positive powers of μ.

For the first integral, since $|\mu| \geq 1$ is the only interested situation, p_j is homogeneous, so

$$\begin{aligned} &\int_{|\xi|\geq|\mu|}\int_G e^{i(x-gx)\cdot\xi} p_j(x,\xi,\mu)dg\, d\xi \\ &= \mu^{m_j+n}\int_{|\xi|\geq 1}\int_G e^{i|\mu|(x-gx)\cdot\xi}(|\mu|^{-n-m_j}\, p_j)(x,\xi,\mu/|\mu|)dg\, d\xi \\ &= \mu^{m_j+n}\int_1^\infty\int_{|\xi'|=1}\int_G e^{it|\mu|(x-gx)\cdot\xi'}(|\mu|^{-n-m_j}\, p_j)(x,t\xi',\mu/|\mu|)dg\, d\xi'\, dt. \end{aligned} \tag{4.66}$$

by the oscillatory integral theorem 4.3.1 and then integrate over t which are termwise convergent, this gives terms (4.57) directly in the estimate.

The third integral break down into integrals of for those of q_k, and R_M which are all homogeneous,

$$\begin{aligned}&\mu^{-k}\int_{1\le|\xi|\le|\mu|}\int_G e^{i(x-gx)\cdot\xi}q_k(x,\xi)dg\;d\xi\\&=\mu^{-k}\int_1^{|\mu|}\tau^{m_j+k+n-1}\int_{S^{n-1}}\int_G e^{i\tau(x-gx)\cdot\xi'}q_k(x,\xi')dg\;d\xi'\;d\tau\end{aligned}\tag{4.67}$$

using polar coordinates in ξ-plane, we get another part that contribute to these logarithm expansions.

For R_M, which can be extended to homogeneous in ξ for all $\xi\neq 0$, with a difference of the second integral type, choose large enough $M>-n-m_j$, so that r_M gives terms just as q_k, except also for non-positive powers of μ.

The conclusion follows from the combination of three integrals.

□

Proposition 4.5.10. *Let $A\in\Psi^k(E,E)\rtimes G$, P be a second order weakly polyhomogeneous, self-adjoint and positive elliptic pseudo-differential operator on E, and let be Γ a sector near negative real axis. Then for $Re(z)>(k+n)/2$, AP^{-s} is trace class, $Trace(AP^{-s})$ is analytic in z on the half plane $Re(z)>(k+n)/2$, extending to a meromorphic function on $\mathbb{C}$ and up to an entire function in $z\in\mathbb{C}$,*

$$\Gamma(z)Tr(AP^{-z})\sim\sum_{j\ge0}\frac{\tilde c_j}{z+\frac{j-k-n}{2}}+\sum_{l\ge0}\left(\sum_{p=0}^{\dim G+n-1}\frac{\tilde c'_{p,q}}{(z-\frac{k+n}{2}+\frac{l}{q})^{p+1}}\right),\tag{4.68}$$

where all the coefficients except those contributed also by the first sum are determined by the symbol.

Proof. By theorem 4.5.8, $A(P+\mu^2)$ is weakly polyhomogeneous, so we apply proposition 4.5.9, for $-\lambda=\mu^2$. Apply the integral transformation in lemma 4.4.2. If A' is smoothing, then by corollary 4.5.5 and proposition 4.5.10

$$Tr(A'(P-\lambda)^{-N})\sim\sum_{j\ge0}c_j(-\lambda)^{-N-j/2}\tag{4.69}$$

which does not contribute to the pure logarithm and their power terms. That is the generalized noncommutative residues are locally computable.

□

4.6 Transversal residue formula

Definition 4.6.1. For a pair of conic neighborhood V_1, V_2 of $T^*_G M$ with $\bar{V_1} \subset V_2$, let

$$\Psi^m_+(E, V_1, V_2) \tag{4.70}$$

be those $P \in \Psi^m(E)$ such that

1. P is positive;
2. $Ess(P) \cap V_1 = \varnothing$;
3. P is G invariant;
4. If $\xi \in T^*M_x \backslash \{0\} \backslash V_2$ and $|\xi| > 1$, then the symbol $\sigma(P)(x, \xi)$ of P is positive definite.

Lemma 4.6.2. *If V_1 and V_2 are small enough, $\Psi^m_+(E, V_1, V_2)$ is nonempty.*

Proof. For any $Y \in \mathfrak{g}$, let

$$Y_E : L^2(E) \to L^2(E) \tag{4.71}$$

defined by the infinitesimal action of $\rho(exp(tY))$ on E, as $t \to 0$. For any $Y \in \mathfrak{g}$, Y_E is a first order differential operator on E.

For an orthonormal basis Y_i, $i = 1, \ldots, \dim \mathfrak{g}$, of $\mathfrak{g}$, let W_A be the differential operator introduced by Atiyah [4]:

$$W_A = 1 - \sum_{i=1}^{\dim \mathfrak{g}} Y^2_{i,E}. \tag{4.72}$$

So the principal symbol of W_A is positive definite on $T^*M \backslash \{0\} \backslash T^*_G M$. Atiyah ([4]) used W_A to prove the existence of the distributional index. Furthermore we can modify W_A so that it achieves the same but also transversally smoothing.

Choose a pair of small conic neighborhood V_1, V_2 of $T^*_G M$. For this V let χ and P_χ be the function and pseudo-differential introduced in example 3.5.2. We assume P_χ is G-invariant, by averaging if necessary. $W'_A = P_\chi W_A (P_\chi)*$ has essential support disjoint with $T^*_G M$, and when V_1 and V_2 are small enough $Q + W'_A$ is positive and elliptic. In particular $W'_A \in \mathcal{K}_G$. $\square$

Now take a G-invariant $Q \in \Psi^2(M)$ and assume it is positively ordered, transversally elliptic and positive. It follows that the principal symbol of Q is positive definite on T^*_G.

Proposition 4.6.3. *Let Q be as above. Then for any $W \in \Psi^2_+(E, V_1, V_2)$, $Q + W$ is elliptic, in addition to the above conditions. Moreover for all $k = 0, 1, \ldots, \dim M + \dim G - 1$ the residues τ_k^{Q+W} are independent of the choice of $W \in \Psi^m_+(E, V_1, V_2)$.*

Proof. When we add W_A to Q we get an operator $Q+W_A$ which is G invariant and elliptic. For K_1 and $K_2 \in \Psi^m_+(E, V_1, V_2)$, we have

$$(Q + K_1 - \lambda)^{-1} - (Q + K_2 - \lambda)^{-1} = (Q - \lambda)^{-1}(K_1 - K_2)(Q - \lambda)^{-1}. \quad (4.73)$$

But the symbol right hand side is in $S^{-\infty,-2}$, so it has no contribution to the residues. $\square$

Definition 4.6.4. For $A \in \Psi^\infty(E) \rtimes G$, Q as above, we define

$$\tau_k^Q(A) = \tau_k^{Q+K}(A) \quad (4.74)$$

for some $K \in \Psi_+(E, V_1, V_2)$ and sufficiently small conic neighborhoods V_1 and V_2.

In fact, more careful examination of the modified operators, we find that they are close to an invariant operator on T^*_G direction tensored with an smoothing operator on the complimentary direction. In fact, Brünning and Heintz [11] used similar techniques on such operators and showed the following theorem. Let m be the dimension of M_0/G. Let (r, V) be any irreducible representation of G, and let χ_r be the character of r:

$$\chi_r(g) = Trace_V(r(g)).$$

Theorem 4.6.5. *Let $P \in \Psi^{2k}(E)$ be a positive, positively ordered, G invariant and transversally elliptic. Let $\nu(\lambda)$ be the multiplicity of r in the eigenspace of P with eigenvalue λ:*

$$N_r(t) = \sum_{\lambda < t} \nu(\lambda). \quad (4.75)$$

Then $N_r(t)$ is finite and as $t \to \infty$

$$N_r(t) \sim \frac{t^{\frac{m}{2k}}}{m(2\pi)^m} \int_{M_0} \frac{1}{vol(G_x)} \int_{T^*_{G,1}M} Tr_{(E_x \otimes V^*)^{G_x}}(\sigma(\xi_1)^{\frac{-m}{2k}} \otimes id_{V^*}) d\xi_1 dx, \tag{4.76}$$

where G_x is the stabilizer at x and dx is the volume form. The error term is $O(t^{\frac{m-1}{2k}} \log t)$.

Let c be the coefficient of $t^{\frac{m}{2k}}$.
By an integral transform:

$$Trace(\rho(\pi^*(\chi_r))P^s) = \int_0^\infty t^z dN_r(t) \tag{4.77}$$

we get

$$Res_{s=-m} Trace(\rho(\pi^*(\chi_r))P^s) = c \tag{4.78}$$

Theorem 4.6.6. *Under the above assumptions, for any Dixmier trace Tr_ω we have*

$$\begin{aligned} &Tr_\omega(\rho(\pi^*(\chi_r))Q^{-m}) \\ &= \frac{1}{m(2\pi)^m} \int_{M_0} \frac{1}{vol(G_x)} \int_{T^*_{G,1}M} Tr_{(E_x \otimes V^*)^{G_x}}(\sigma(\xi_1)^{\frac{-m}{2k}} \otimes id_{V^*}) d\xi_1 dx. \end{aligned} \tag{4.79}$$

In particular, all $\tau_k^Q = 0$ for $k \geq 1$.

5 The index of a transversally elliptic operator

Let $P \in \Psi^k(E, F)$, $P : \Gamma(E) \to \Gamma(F)$, be a transversally elliptic pseudo-differential operator relative to G action on E and F. Let

$$\pi_P : L^2(E, M) \to ker(P) \tag{5.1}$$

be the projection to the kernel of P. In this section we define the index as a distribution on G.

5.1 Definition of the index

Lemma 5.1.1. *For any $u \in L^2(E)$, $WF(\pi_P(u)) \subset char(P)$.*

Proof. By the regularity theorem 2.3.18,

$$WF(\pi_P(u)) \subset WF(P(\pi_P(u)) \cup char(P). \tag{5.2}$$

But $P(\pi_P(u)) = (P\pi_P)(u) = 0$ and so it has empty wave front set.

$\square$

Theorem 5.1.2. *Let P be a transversally elliptic operator. For any $\phi \in \mathcal{A}$, $\rho(\phi)\pi_P$ is a smoothing operator. In particular, for any $f \in C_c^\infty(G)$, $\rho(f)\pi_P$ is trace class.*

Proof. $\rho(f) = \rho(\pi_2^*(f))$ is a special case of $\rho(\phi)$. By Lemma 5.1.1, Lemma 3.4.2, Theorem 2.2.10,for any $u \in L^2(E)$,

$$WF(\rho(\phi)\pi_P u) \subset WF'(\rho(\phi)) \circ WF(\pi_P u).$$

Since P is transversally elliptic, $(T_G^*M) \cap char(P) = \varnothing$, the above composition is empty. So $\rho(\phi)\pi_P$ is a smoothing operator, in particular, trace class. $\square$

Now we are ready to introduce the definition of the index of a transversally elliptic operator. The following is equivalent to Atiyah's original definition [4]. Atiyah pointed out in [4] the wave front set approach by Hörmander(see, for example, [30]) will show that the index below also makes sense as a distribution on the Lie group even for non-compact Lie groups.

Definition 5.1.3. The index of a transversally elliptic operator P is defined to be the distribution on G such that for any $f \in C_c^\infty(G)$,

$$index^G(P)(f) = Trace(\rho(f)\pi_P) - Trace(\rho(f)\pi_{P^*}). \tag{5.3}$$

We observe that it is possible to define the index without the G-invariant condition (see [26] and [27]). Since P is G-invariant under our assumption, the index is a central distribution on G.

Definition 5.1.3 naturally extends to a distribution on the groupoid $\mathcal{G}$ induced by the action of G on M.

Definition 5.1.4. We define the local index density of a transversally elliptic operator P as the distribution on $\mathcal{G}_1 = M \times G$ such that for any $\phi \in \mathcal{A} = C_c^\infty(M \times G)$,

$$index^{\mathcal{G}}(P)(\phi) = Trace(\rho(\phi)\pi_P) - Trace(\rho(\phi)\pi_{P^*}). \tag{5.4}$$

When G is the trivial group, then $index^{\mathcal{G}}$ is a smooth function on M – the index density as in the heat kernel proof of the index theorem for classical operators – and its integral on M is the index of the operator P. Although the index is a topological invariant, the index density is not.

5.2 K-homology of the algebra $\mathcal{A}$

We recall some definitions. A p-summable pre-Fredholm module over $\mathcal{A}$ is a pair $(\mathcal{H}, F)$ where

(1) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a $\mathbb{Z}_2$-graded Hilbert space with grading $\epsilon = 1_{\mathcal{H}_+} \oplus (-1_{\mathcal{H}_-})$, which is a $\mathbb{Z}_2$-graded left $\mathcal{A}$-module,

(2) $F \in B(\mathcal{H})$, $F\epsilon = -\epsilon F$, $\phi(F^2 - 1)$ is compact for any $\phi \in \mathcal{A}$,

(3) for any $\phi \in \mathcal{A}$, $[F, \phi] \in \mathcal{L}^p(\mathcal{H})$, the p-Schatten ideal of compact operators.

A p-summable pre-Fredholm $\mathcal{A}$-module is called a *p-summable Fredholm $\mathcal{A}$-module* if in addition $F^2 = 1$.

Let

$$P : \Gamma^\infty(M, E) \to \Gamma^\infty(M, F), \tag{5.5}$$

be a G-invariant transversally elliptic pseudo-differential operator of order 0. P has a transversal parametrix Q, which can be assumed to be G-invariant as well (by averaging on G). Let $\mathcal{H} = L^2(E) \oplus L^2(F)$ with grading $\epsilon = 1_{L^2(E)} \oplus (-1_{L^2(F)})$, it is a graded $\mathcal{A}$-module through the action $\rho_E \oplus \rho_F$. Let

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix} \tag{5.6}$$

Theorem 5.2.1. *Let P in $\Psi^0(M; E, F)$ be G-invariant and transversally elliptic. Then the pre-Fredholm module $(\mathcal{H}, F)$ introduced as above is p-summable for all $p > \dim M$.*

Proof. Condition (1) is obvious. (2) follows from pseudo-local property of pseudo-differential operators and G-invariance of P and Q.

For (3), we have

$$[F, \phi] \in \Psi^{-1})G(E \oplus F, E \oplus F) \tag{5.7}$$

and by the $*$-algebra properties of $\Psi_G^{-1}(E \oplus F, E \oplus F)$

$$|[F, \phi]|^{n+1} \in \Psi_G^{-n-1}(E \oplus F, E \oplus F) \subset \mathcal{L}^1\mathcal{H}. \tag{5.8}$$

$\square$

We now recall the standard process to transform a pre-Fredholm $\mathcal{A}$-module into a Fredholm $\mathcal{A}$-module, preserving p-summability (for details see [13], Appendix II of part I).

Given a pre-Fredholm $\mathcal{A}$ module $(\mathcal{H}, F)$, let $\tilde{\mathcal{H}} = \mathcal{H}\hat{\otimes}\mathcal{C}$ be the graded tensor product of $\mathcal{H}$ with a $1+1$ dimensional graded Hilbert space $\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-$, with $\mathcal{C}_\pm = \mathbb{C}$. Then $\tilde{\mathcal{H}}_+ = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\tilde{\mathcal{H}}_- = \mathcal{H}_- \oplus \mathcal{H}_+$. The $\mathcal{A}$-module structure on $\tilde{\mathcal{H}}$ is given by

$$\tilde{\rho}(\phi)(\xi\tilde{\otimes}\eta) = (\rho(\phi)\xi)\hat{\otimes}(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\eta), \tag{5.9}$$

for any $\phi \in \mathcal{A}$, $\xi \in \mathcal{H}$, $\eta \in \mathcal{C}$ ($\mathcal{A}$ acts only non-trivially on $\tilde{\mathcal{H}}_+$ as in $\mathcal{H}$). Since F is of odd order, $\epsilon F = -F\epsilon$, F is alway of the form

$$F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}, \tag{5.10}$$

so we define

$$\tilde{F} = \begin{bmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{bmatrix} \tag{5.11}$$

where

$$\tilde{P} = \begin{bmatrix} P & 1 - PQ \\ 1 - QP & (QP - 2)Q \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} (2 - QP)Q & 1 - QP \\ 1 - PQ & -P \end{bmatrix}. \tag{5.12}$$

Proposition 5.2.2. *For a pre-Fredholm $\mathcal{A}$-module $(\mathcal{H}, F)$, $(\tilde{\mathcal{H}}, \tilde{F})$ is a Fredholm $\mathcal{A}$-module, and there exists a pre-Fredholm $\mathcal{A}$-module $(\mathcal{H}_0, 0)$ with zero $\mathcal{A}$-action such that*

(1) $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}_0$,

(2) for any $\phi \in \mathcal{A}$, $\phi(\tilde{F} - F \oplus 0)$ is compact.

Moreover, if $(\mathcal{H}, F)$ is p-summable, so is $(\tilde{\mathcal{H}}, \tilde{F})$.

In [13] (Part I, section 1), Connes showed that there is a trace $\tau : \mathcal{A} \to \mathbb{C}$,

$$\tau(\phi) = \frac{1}{2} Trace(\epsilon F[F, \phi]) \tag{5.13}$$

which gives the index map $K_0(\mathcal{A}) \to \mathbb{Z}$,

$$index F_e^+ = (\tau \otimes Trace)(e) \tag{5.14}$$

for any projection e ($e = e^* = e^2$) in the finite matrix algebra $M_q(\tilde{\mathcal{A}})$ for arbitrary q.

Definition 5.2.3. (Connes, [13]) The trace τ (denoted by $char(\tilde{H}, \tilde{F})$) is called the Connes character of the 1-summable Fredholm $\mathcal{A}$-module $(\tilde{\mathcal{H}}, \tilde{F})$.

Next we show that the character τ of $(\tilde{\mathcal{H}}, \tilde{F})$ is just like the local index density. This fact allows us to get transversal index formula by computing the Connes character.

Proposition 5.2.4.

$$index^G(P) = \pi_* char(\tilde{H}, \tilde{F}), \tag{5.15}$$

where $(\pi)_* = (\pi_G)_*$ *is the push-forward by the projection* $\pi = \pi_G : M \times G \to G$.

Proof. In fact, for a G-invariant transversally elliptic pseudo-differential operator P, let G be the Green operator on $L^2(F)$, i.e.,

$$GP = 1 - \pi_P, \quad PG = 1 - \pi_{P^*}. \tag{5.16}$$

So the local index density can be written as

$$index^G(f) = Trace(\rho(\pi^* f)(1 - GP)) - Trace(\rho(\pi^* f)(1 - PG)). \tag{5.17}$$

For a transversal parametrix Q of P, let

$$\tau'(\rho(\pi^* f)) = Trace((\rho(\pi^* f)(1 - QP)) - Trace(\rho(\pi^* f)(1 - PQ)) \tag{5.18}$$

which is well defined by proposition 3.6.1. First, we show that

$$\pi_* \tau' = index^G(P). \tag{5.19}$$

From

$$\pi_P = (1 - QP)\pi_P, \quad \pi_{P^*} = \pi_{P^*}(1 - PQ), \tag{5.20}$$

we have

$$\begin{aligned} index^G(P)(f) =& Trace(\rho(\pi^* f)\pi_P) - Trace(\rho(\pi^* f)\pi_{P^*}) \\ =& Trace(\rho(\pi^* f)(1 - QP)\pi_P) - Trace(\rho(\pi^* f)\pi_{P^*}(1 - PQ)) \\ =& \tau'(\pi^* f) - Trace(\rho(\pi^* f)(1 - QP)GP) \\ & \quad + Trace(\rho(\pi^* f)PG(1 - PQ)) \\ =& \tau'(\pi^* f). \end{aligned} \tag{5.21}$$

The last equality in the above equation holds since the last two terms are

$$\pm Trace(\rho(\pi^*(f))G(P - PQP)). \tag{5.22}$$

Repeating the above process we conclude that for any integer n,

$$index^G(P)(\pi^* f) = Trace(\rho(\pi^* f)(1 - QP)^n) - Trace(\rho(\pi^* f)(1 - PQ)^n). \tag{5.23}$$

Direct computation shows that:

$$\begin{aligned} \tau(\pi^* f) =& \frac{1}{2} Trace(\begin{bmatrix} \rho_E(\pi^* f) \oplus 0_F & 0 \\ 0 & -\rho_F(\pi^* f) \oplus 0_E \end{bmatrix} \\ & - \begin{bmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{bmatrix} \begin{bmatrix} \rho_E(\pi^* f) \oplus 0_F & 0 \\ 0 & -\rho_F(\pi^* f) \oplus 0_E \end{bmatrix} \begin{bmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{bmatrix}) \\ =& Trace(\rho_E(\pi^* f)(1 - QP)^2) - Trace(\rho_F(\pi^* f)(1 - PQ)^2) \\ =& \tau'(\pi^* f). \end{aligned} \tag{5.24}$$

The constant factor $1/2$ is not essential, it depends on our particular choice of the way which a pre-Fredholm is transformed into a Fredholm module.

For p-summable ($p > 1$) Fredholm module, using some $n > p$ in (5.23) and (5.24). □

5.3 The spectral triple in transversally elliptic case

In this section we construct a spectral triple associated to a transversally elliptic pseudo-differential operator P. We discuss only the even case to

simplify our argument, the odd case is similar. First we may assume it is of order 1 as a pseudo-differential operator.

because when necessary we may multiply D with an appropriate power of $1+\Delta$, this operation is an isomorphism of Sobolev spaces, hence it does not alter the index of D.

Now let $\mathcal{H} = L^2E \oplus L^2F$, $\epsilon = 1_E \oplus (-1)_F$, and $\mathcal{A}$ acts on $\mathcal{H}$ by $\rho_E \oplus \rho_F$ as before. Let D be an operator of odd grading,

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}. \tag{5.25}$$

We are interested in the case when D is symmetric, which is equivalent to $D_- = (D_+)^*$. We now show that if D is symmetric then D is essentially self-adjoint.

For a symmetric pseudo-differential operator

$$D : \Gamma_c^\infty(M, E \oplus F) \to \Gamma^\infty(M, E \oplus F), \tag{5.26}$$

we have an extension

$$D' : \Gamma^{-\infty}(M, E \oplus F) \to \Gamma_c^{-\infty}(M, E \oplus F). \tag{5.27}$$

Proposition 5.3.1. *If $D \in \Psi(E \oplus F)$ is transversally elliptic, symmetric and has positive order, then as an unbounded operator on $\mathcal{H}$, defined on the domain of smooth sections, D is essentially self-adjoint.*

Kordyukov [27]) proved a more general statement.

Proof. To show D is essentially self-adjoint we need only to show that (see, for instance, theorem 26.1 of [34])

$$Ker(D^* \pm iI) \subset Dom(\bar{D}). \tag{5.28}$$

First we recall the proof ([34]) that

$$Dom(D^*) = \{s \in \mathcal{H}; D's \in \mathcal{H}\}. \tag{5.29}$$

We denote by W the right hand side of (5.29). For all $u \in \Gamma^\infty$, $s \in \mathcal{H}$, by definition

$$\langle Du, s \rangle = \langle u, D's \rangle . \tag{5.30}$$

This implies $W \subset Dom(D^*)$ and $D'|_W = D^*|_W$. If $u \in Dom(D^*)$, then there is a $w \in \mathcal{H}$ such that for any $v \in \Gamma^\infty$,

$$\langle u, Dv \rangle = \langle D^* u, v \rangle = \langle w, v \rangle . \tag{5.31}$$

But this implies $D'u = w$ so $w \in \mathcal{H}$. In other words, $u \in W$.

Next we show that

$$S = \{ s \in W \,|\forall \phi \in \mathcal{A}\ \rho(\phi)s \in \Gamma^\infty \} \tag{5.32}$$

is contained in $Dom(\bar{D})$.

Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of bump functions on G converging to the delta distribution δ at the identity on G. For $s \in W$, we have $\rho(\pi^*(f_n))s \to s$ in $\mathcal{H}$ and

$$\rho(\pi^*(f_n))D's = D'\rho(\pi^*(f_n))s \to D's \tag{5.33}$$

in any $\mathcal{H}$. Recall that (the graph of) $\bar{D}$ is defined by the closure of its graph. So we have $s \in Dom\bar{D}$.

Now it is clear to prove (5.28) we need only to show $Ker(D^* \pm i) \subset S$. The proof for the two cases are essentially the same. If $(D^* + i)s = 0$, then $D's \in \mathcal{H}$ and $(D' + i)s = 0$. Since D' is of positive order, $D' + i$ and D' have the same principal symbol. So there is an G-invariant Q that is a transversal parametrix for $D + i$: $\rho(\phi)[1 - Q(D + i)]$ is smoothing. Thus

$$\rho(\phi)s = \rho(\phi)(1 - Q(D + i))s + 0 \in \Gamma^\infty, \tag{5.34}$$

which says $s \in S$.

□

From now on we may assume D is self-adjoint, replacing D by its closure when necessary.

Lemma 5.3.2. *For any nonzero real number λ, $\rho(\phi)(D - \lambda i)^{-1}$ and $(D - \lambda i)^{-1}\rho(\phi)$ are compact operators for all $\phi \in \mathcal{A}$.*

Proof. Let Q_λ be a transversal parametrix for $D - \lambda i$. As discussed above, it might not necessarily have negative order, but it can be chosen so by a cutoff on the symbol in a conic neighborhood of $T^*_G M$. So $1 - Q_\lambda(D - \lambda i) = K$ and $1 - (D - \lambda i)Q_\lambda = K'$. Apply the inverse to the right hand side of the first parametrix formula, we have

$$(D - \lambda i)^{-1} - Q_\lambda = K(D - \lambda i)^{-1}. \tag{5.35}$$

Q_λ is bounded since it is a pseudo-differential operator of order zero. It suffices to show that $\rho(\phi)K(D-\lambda i)^{-1}$ is smoothing. $(D-\lambda i)^{-1}$ is a bounded operator from H^s to H^{s+1} so its composition with a smoothing operator is still smoothing. $\square$

Proposition 5.3.3. *$(\mathcal{A}, \mathcal{H}, D)$ is a $\dim M^+$-summable spectral triple.*

Proof. Since D commutes with $\rho(g)$, For any $\phi \in \mathcal{A}$,

$$[D, \phi] = \int_G [D, \phi(x, g)]\rho(g)d\mu(g) \tag{5.36}$$

so it is compact, as $[D, \phi(x, g)]$ is a pseudo-differential operator of negative order. Lemma 5.3.2 shows $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. For any $\phi \in \mathcal{A}$, and $\lambda \in \mathbb{C}\backslash\mathbb{R}$,

$$\phi(\lambda - |D|)^{-1} = \phi Q_\lambda + \phi K_\lambda(\lambda - |D|)^{-1} \in \mathcal{L}^{(dim(M),\infty)} \tag{5.37}$$

since the transversal parametrix can be chosen to be a pseudo-differential operator of order -1, ϕK_λ is trace class and $(\lambda - |D|)^{-1}$ is bounded. $\square$

Lemma 5.3.4. *When $|D|$ has a scalar principal symbol, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is regular.*

Proof. We need to show

$$\mathcal{A} \cup [D, \mathcal{A}] \subset Dom^\infty(\delta), \tag{5.38}$$

and in fact we will show that

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \Psi^0(E, E) \rtimes G \subset Dom^\infty(\delta), \tag{5.39}$$

and δ preserves $\Psi^0(E, E) \rtimes G$.

Elements in $\mathcal{A}$ have scalar symbols, so elements $[D, \mathcal{A}] \in \Psi^1(E, E) \rtimes G$ have vanishing principal symbols, which implies $[D, \mathcal{A}] \in \Psi^0(E, E) \rtimes G$.

In general, since $|D|$ has scalar symbols of degree 1, the commutator with any classical pseudo-differential operator has order, at most zero. And since $|D|$ is G-invariant, the composition with $\rho(g)$ has no effect. $\square$

5.4 Asymptotic spectral analysis for spectral triples

With an extra parameter on pseudo-differential operators, that is, for families of pseudo-differential operators, the above properties still hold.

By lemma 5.3.4 we have a regular spectral triple when $|D|$ has scalar principal symbols. As we showed in section 5.1, we need to study the poles of

$$\zeta_A(z) = Tr(A\,|D|^{-2z}) \tag{5.40}$$

where $A \in \mathcal{A}_D \subset \Psi^0(E,E) \rtimes G$. For the purpose of estimation, we will relax the condition on A, only assuming $A \in \Psi^0(E,E)$.

5.5 Connes-Chern character in the periodic cyclic cohomology of $\mathcal{A}$

As we have seen in the previous discussion, the computation of the index for transversally elliptic pseudo-differential operators amounts to the computation of the Connes character of a finitely summable Fredholm module. The Connes character evolved into its new version that takes values in cyclic cohomology.

For a pre-C^*-algebra A, and a p-summable Fredholm A-module (H,F), let

$$Tr'(T) = \frac{1}{2}Trace(\epsilon F[F,T]) \tag{5.41}$$

as the Connes character $(\tau(\pi^* f) = Tr'(\pi^* f))$. The Connes-Chern character in the periodic cyclic cohomology

$$ch^*(H,F) \in HP^*(A) \tag{5.42}$$

is defined to be

$$ch^*(H,F)(a^0,\dots,a^n) = (-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1)Tr'\left(a^0[F,a^1]\cdots[F,a^n]\right) \tag{5.43}$$

for n even and

$$ch^*(H,F)(a^0,\dots,a^n) = \sqrt{2i}(-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1)Tr'\left(a^0[F,a^1]\cdots[F,a^n]\right) \tag{5.44}$$

for n odd (Here Γ is the Gamma function).

As before, let $e \in M_q(\tilde{\mathcal{A}})$ be a projection (which is equivalent to a finitely generated projective module on $\mathcal{A}$ in the K-theory $K_*(\mathcal{A})$ for operator algebras). There is a well understood Chern character (see [14],[10]) from $K_*(\mathcal{A})$ to the periodic cyclic homology of $\mathcal{A}$:

$$ch_*(e) \in HP_*(\mathcal{A}). \tag{5.45}$$

The Connes character can be viewed the dual of this Chern character. The Connes-Chern character gives the index formula (for example, in the even case) in the following fashion. For an element $[e] \in K(A)$, then the index of F_e^+ the twisted operator F^+ by the the projection e is

$$Index F_e^+ = \langle ch^*(H, F), ch_*(e) \rangle. \tag{5.46}$$

In conclusion, to find the index of P and its twisted versions, we may compute the Connes-Chern character of a Fredholm module F associated with it, in the periodic cyclic cohomology.

5.6 The Connes-Moscovici local index formula

First, we briefly recall the definitions and results from [12] (also see [14],[18]).

Let $\mathcal{A}$ be a $*$-algebra, which is a dense $*$-subalgebra of a pre-C^* algebra A.

Definition 5.6.1. *An spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where

(1) $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a $\mathbb{Z}_2$-graded Hilbert space and left $\mathcal{A}$-module, with grading ϵ;

(2) D is an unbounded self-adjoint operator on $\mathcal{H}$ such that $D\epsilon = -\epsilon D$;

(3) for all $a \in \mathcal{A}$, $[D, a] \in B(\mathcal{H})$;

(4) for all $a \in \mathcal{A}$, $a(1 + D^*D)^{-1}$ is compact.

An odd spectral triple over A is similarly defined except without grading and grading related conditions in the above.

Starting with a spectral triple, by the observation in [7] by Baaj and Julg, the following assignment

$$D \mapsto D(1 + D^*D)^{-1/2} \tag{5.47}$$

determines a pre-Fredholm module. In fact it is shown in [7] that all K-homology classes of a pre-C^* algebra can be obtained this way. We may

switch to the computation of the Connes-Chern character of a spectral triple for the following reasons: the index of D is preserved in the Baaj-Julg assignment; the Connes-Chern character is an invariant of the K-homology class; and when D is a pseudo-differential operator, so is $D(1+D^*D)^{-1/2}$ whose symbol can be computed in terms of the symbol of D.

Let F and $|D|$ be the elements of the polar decomposition of D:

$$D = F\,|D|\,, \tag{5.48}$$

where $F = signD$ is unitary and $|D| = (D^2)^{1/2}$ is positive. $(\mathcal{H}, F)$ is a bounded pre-Fredholm module, representing the same class in K-homology determined by the spectral triple. For what we do, the invertibility of $|D|$ is not so important, a small shift of its spectrum by a positive number will not affect the asymptotic behavior of the spectrum. As we have showed, for our purpose, $|D|^{-1}$ only need to be well defined up to a smoothing operator. For example the Green operator works fine:

$$|D|\,G = G\,|D| = 1 - \pi_{\ker D}. \tag{5.49}$$

Definition 5.6.2. For some $p \geq 1$, a p^+-summable spectral triple is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ such that for any $\phi \in \mathcal{A}$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$,

$$\phi(\lambda - |D|)^{-1} \in \mathcal{L}^{p+}(\mathcal{H}) = \mathcal{L}^{(p,\infty)}(\mathcal{H}), \tag{5.50}$$

where for $p > 1$, $\mathcal{L}^{(p,\infty)}$ is the ideal of $B(\mathcal{H})$ consisting of those compact operators T whose n-th characteristic value $\mu_n(|T|) = min\{\,\|\,|T|\,|_{E^\perp}\|; \dim E = n\}$ satisfies

$$\mu_n(|T|) = O(n^{-1/p}). \tag{5.51}$$

When $p = 1$, this is a sufficient condition for $T \in \mathcal{L}^{(1,\infty)}$, but usually this is satisfied.

Let δ be the derivation operator $ad(|D|)$:

$$\delta(T) = ad(|D|)(T) = [|D|\,, T] \tag{5.52}$$

defined on the bounded operators $B(\mathcal{H})$ and takes values as unbounded operators on $\mathcal{H}$. Let $Dom(\delta) \subset B(\mathcal{H})$ be the domain of δ; that is, $A \in B(\mathcal{H})$ is in $Dom(\delta)$ if and only if $[|D|\,, A]$ extends to a bounded operator on $\mathcal{H}$. And let

$$Dom^\infty(\delta) = \bigcap_{k\geq 1} Dom(\delta^k). \tag{5.53}$$

Definition 5.6.3. A p^+-summable spectral triple is regular if

$$\mathcal{A} \cup [D, \mathcal{A}] \subset Dom^\infty(\delta) \tag{5.54}$$

For a n^+-summable spectral triple (A, H, D) which satisfies

$$\mathcal{A} \cup [D, \mathcal{A}] \subset Dom(\delta^2), \tag{5.55}$$

Connes character formula says the n-cocycle in Hochschild cohomology of $\mathcal{A}$ is, for $a_i \in A$,

$$\phi_\omega(a^0, \ldots, a^n) = \lambda_n Tr_\omega(\epsilon a^0 [D, a^1] \cdots [D, a^n] \, |D|^{-n}), \tag{5.56}$$

where λ_n is a universal constant, and Tr_ω is the Dixmier trace (see [14] IV.2. for details). In the odd case the same is true with $\epsilon = id$.

Connes-Moscovici [12] went further to solve the general problem for local character formula for spectral triples with discrete dimension spectrums. We now to introduce more notations. Let $\mathcal{A}_D$ be a subspace of $B(\mathcal{H})$ generated by the following operations: for any operators $A \in \mathcal{A}$,

$$dA = [D, A], \ \nabla(A) = [D^2, A], \ A^{(k)} = \nabla^k(A)$$

are in $\mathcal{A}_D$, moreover the operators

$$P(a^0, a^1, \ldots, a^n) = a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)}$$

are in $\mathcal{A}_D$, where $a^0, \ldots, a^n \in \mathcal{A}$, acting on $\mathcal{H}$ by ρ.

To apply our result for the residues the Connes-Moscovici theorems, we first list the following criterions in our specific spectral triple.

Criterions We assume that $(\mathcal{A}, \mathcal{H}, D)$ is (1) p^+-summable for some $p > 1$, (2) regular, (3) with discrete dimension spectrum, (4) for $P \in \mathcal{A}_D$, the zeta function

$$\zeta_{P,D}(z) = Trace(P \, |D|^{-2z})$$

is at least defined and analytic for $Re(z) > k + p$. In this case

$$\tau_q(P) = \tau_q^{|D|}(P) = Res_{z=0} z^q \zeta_{P,D}. \tag{5.57}$$

(5) Only finite many of τ_q, $q = 0, \ldots$ are nonzero in general.

We have seeing that our spectral $(\mathcal{A}, \mathcal{H}, D)$ for the transversally elliptic case is regular, p^+-summable, with discrete dimension spectrum, and the poles of $\zeta_{P,D}$ have multiplicities not exceeding a fixed number. Therefore we have the following.

Theorem 5.6.4. *(Connes-Moscovici) a) The following formula defines an even cocycle in (b, B) bicomplex of $\mathcal{A}$:*

$$\begin{aligned} \phi_0(a_0) &= \tau_{-1}(\gamma a_0) \\ \phi_{2m}(a_0, \ldots, a_{2m}) &= \sum_{k \in \mathbb{N}^{2m}, q \geq 0} c_{2m,k,q} \\ &\qquad \tau_q(\gamma a_0 (da_1)^{(k_1)} \ldots (da_{2m})^{k_{2m}} |D|^{-2|k|-2m}) \end{aligned} \tag{5.58}$$

for $m > 0$, where $c_{2m,k,q}$ are universal constants given by

$$c_{2m,k,q} = \frac{(-1)^{|k|}}{k!\tilde{k}!} \sigma_q(|k| + m) \tag{5.59}$$

where $k! = k_1! \ldots k_{2m}!$, $\tilde{k}! = (k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \ldots + k_{2m} + 2m)$, and $\sigma_q(N)$ is the q-th elementary polynomial of the set $\{1, 2, \ldots, N-1\}$.

(b) The cohomology class of the cocycle $(\phi_{2m})_{m \geq 0}$ in $HC^{ev}(\mathcal{A})$ coincides with the Connes-Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.

In the odd case, the Connes-Chern character is computed similarly. Suppose we have an odd spectral triple still called $(\mathcal{A}, \mathcal{H}, D)$.

Theorem 5.6.5. *(Connes-Moscovici) a) The following formula defines an odd cocycle in (b, B) bicomplex of $\mathcal{A}$:*

$$\begin{aligned} \phi_{2m+1}(a_0, \ldots, a_{2m+1}) &= \sqrt{2i} \sum_{k \in \mathbb{N}^{2m+1}, q \geq 0} c_{2m+1,k,q} \\ &\qquad \tau_q(a_0 (da_1)^{(k_1)} \ldots (da_{2m+1})^{k_{2m+1}} |D|^{-2|k|-2m-1}) \end{aligned} \tag{5.60}$$

where $c_{2m+1,k,q}$ are universal constants given by

$$c_{2m+1,k,q} = \frac{(-1)^{|k|}}{k!\tilde{k}!q!} \Gamma^{(q)}(|k| + m + \frac{1}{2}) \tag{5.61}$$

where $\Gamma^{(q)}$ is the q-th derivative of the Gamma function.

(b) The cohomology class of the cocycle $(\phi_{2m+1})_{m \geq 0}$ in $HC^{od}(\mathcal{A})$ coincides with the Connes-Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.

Remark 5.6.6. As a sample, where G is the trivial group. This is the commutative case since $\mathcal{A} = C^\infty(M)$. D is an elliptic pseudo-differential operator

with scalar principal symbol on a hermitian bundle E over a smooth manifold M. There exists such geometric operator D, since we can choose a space-time oriented pseudo-Riemannian manifold with a compact Lie group acting by isometries, assume it is spin, then the natural Dirac operator is obviously not elliptic. But if there is no isotropic (null) directions along the orbit, then the Dirac operator is transversally elliptic relative to the group action (see [8]). The spectral triple is regular, $\dim M^+$-summable, and with discrete dimension spectrum. Since the zeta functions involved have at most simple poles, only τ_0 is possibly nonzero. τ_0 is up to constant factors the Wodzicki residue, which is computable in terms of symbols of the operators involved.

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